Multiorder multidimensional systems: Computation of the transfer function using the DFT

A Thesis
Presented to
The Academic Faculty

by

Marinos Theodorou Michael

In Partial Fulfillment
of the Requirements for the Degree
Master of Science in Computer Science

Department of Computer Science
Montclair State University
May 2006

MONTCLAIR STATE UNIVERSITY

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Professor Robert S. Prezant
Dean of College of Science and Mathematics

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(date)

Thesis Committee:

Professor George E. Antoniou
Thesis Sponsor

Professor Angel Gutierrez
Committee Member

Assistant Professor Stefan A. Robila
Committee Member

Professor Dorothy Deremer
Department Chair

To my family and friends,

for their continued support

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In this thesis the discrete Fourier transform is used for determining the coefficients of the determinantal polynomial and the coefficients of the adjoint polynomial matrix for four systems/models. The main contribution of this thesis include:

- A method for computing the transfer function of a second–order 2D system using the DFT.
- A method for computing the transfer function of a generalized second–order 2D system using the DFT.
- The second–order 2D systems was extended to k–order n–dimensions.
- The generalized second order 2D system was extended to generalized k–order n–dimensions.
- A method for computing the transfer function of a k–order n–dimensional system using the DFT.
- A method for computing the transfer function of a generalized k–order n–dimensional system using the DFT.

CHAPTER 1

INTRODUCTION

Multidimensional (nD) digital systems have received, in recent decades, considerable research interest because of the advancement of digital technology. One Dimensional (1D) systems find wide applications in speech processing (wired and wireless), audio systems, data and computer communications etc. [1], [10], [30], [31], [32]. Systems that are characterized by two independent variables propagating information in two directions, in systems theory, or by two delay elements z_1^{-1}, z_2^{-1}, in circuit theory, require a different approach for the analysis since many 1D techniques can't be extended to 2D [7], [13], [20]. It is noted that systems with multiple delays are used today in electronic industry. The multiple delay elements are considered to be as "scheduling devices", implemented with house software ($Xilinx^{TM}$) over $MATLAB^{TM}$ and hardware with the well known D–type Flip Flops [14], [35]. Applications of Multidimensional systems can be found in digital image processing, seismology, geophysics, remote sensing, robotics, medicine, underwater acoustics, metal–rolling, long-wall coal cutting, etc. [10], [16] ,[20], [21], [25], [26], [34], [36], [37].

Multidimensional systems, like 1D, can be represented with a polynomial transfer function:

$$T_1(z_1, z_2, z_3, \ldots, z_n) = \frac{N(z_1, z_2, z_3, \ldots, z_n)}{D(z_1, z_2, z_3, \ldots, z_n)}$$

or using classical state space like structures, for example:

$$\begin{aligned} \dot{\mathbf{x}}(i_1, i_2, \ldots, i_n) &= \mathbf{A}\mathbf{x}(i_1, i_2, \ldots, i_n) + \mathbf{b}u(i_1, i_2, \ldots, i_n) \\ y(i_1, i_2, \ldots, i_n) &= \mathbf{c}'\mathbf{x}(i_1, i_2, \ldots, i_n) + du(i_1, i_2, \ldots, i_n) \end{aligned}$$

The classical state equation $\dot{\mathbf{x}}(i_1, i_2, \ldots, i_n)$ and output equation $y(i_1, i_2, \ldots, i_n)$, represent the system dynamics and the input-output behavior of the system. The model is linear,

shift invariant and causal [20].

State space based techniques play a very crucial role in the analysis, synthesis and implementation of multidimensional systems.

In this thesis, new system–models for representing k–order n–dimensional systems are presented for the regular and generalized structures. Using the proposed models, computer implementable algorithms are proposed, using the discrete Fourier transform (DFT), for the computation of the transfer function for the following models:

- Second–order 2D system
  - George E. Antoniou, Marinos Michael, *Computing the transfer function for second–order 2D systems*, IEEE International Symposium on Circuits and Systems, Proceedings 237-240 III, Vancouver, 2004.
  - George E. Antoniou, Marinos T. Michael, *Second–order two–dimensional systems: Computing the transfer function*, Journal of Electrical Engineering, Vol. 55, No. 11–12, pp. 296–300, 2004.
- Generalized second–order 2D system
  - George Antoniou, Marinos Michael, *On the generalized second–order 2D System/Model*, International Conference on Circuits and Systems and Computers, Proceedings, pp. 487-490, Athens, Greece, 2004.
  - George E. Antoniou, Marinos T. Michael, *Generalized second–order two – dimensional systems: Computing the transfer function*, Journal of the Circuits, Systems and Computers, Vol. 14 , No. 6 , pp. 1063–1071, 2005.
- k–order n–D system
  - George E. Antoniou, Marinos Michael, *n–Dimensional k–Order systems: Transfer function computation*, International Symposium on Signals Circuits and Systems, Proceedings IEEE Xplore, pp. 183–186, Iasi, Romania, 2005.

- Generalized k–order n–D system

  - George E. Antoniou, Marinos Michael, *Generalized n–Dimensional k–Order systems: Transfer function computation*, IEEE International Symposium on Signals Circuits and Systems, Proceedings, pp. 2299–2302, Cobe, Japan, 2005.

The new system-models are logical extensions for multidimensional systems. The particular models are extension of the 2D Fornasini-Marchesini [11] and the second–order two–dimensional models [3] to n-Dimensions and k-Order. For more two–dimensional (2D) second–order system structures, the reader can refer to [3] . The proposed extension to k–order systems, will increase the order of the numerator and denominator polynomials of the nD transfer function and at the same time will represent systems in more than one dimension. It is noted that the presented model can realize a transfer function with lower matrix-dimensions than the classical models.

1D systems/models with order more that one have been used in the past for Robost pole placement, Model reduction, Real time block recursive parameter estimation, Leverrier and extension of Fadeevs's method to polynomial matrices, in electric power analysis and analysis of flexible beams [8], [9], [17], [19], [27], [29].

In this thesis the DFT is used for determining the coefficients of the determinantal polynomial and the coefficients of the adjoint polynomial matrix for four systems/models.The computational speed of the proposed method can be improved by using fast Fourier transform techniques.

The DFT has been used for the evaluation of the transfer function coefficients for linear, generalized (singular), and multidimensional state space systems [2], [4], [22], [33]. Other methods that could have been used are the Leverrier–Fadeeva, and Vandermonde matrices [6], [18], [28].

The main contribution of this thesis include:

- A method for computing the transfer function of a second-order 2D system using the DFT.

- A method for computing the transfer function of a generalized second-order 2D system using the DFT.
- The second–order 2D systems was extended to k–order n–dimensions.
- The generalized second order 2D system was extended to generalized k–order n–dimensions.
- A method for computing the transfer function of a k–order n–dimensional system using the DFT.
- A method for computing the transfer function of a generalized k–order n–dimensional system using the DFT.

The thesis is organized as follows: Chapter 1 is the introduction. Chapter 2 contains some background information. Chapter 3 explains a method for computing the transfer function of a second-order 2D system using the DFT. Chapter 4 demonstrates a method for computing the transfer function of a generalized second-order 2D system using the DFT. Chapter 5 has the extended k–order n–dimensional system and a method for computing the transfer function using the DFT. Chapter 6 presents the extended generalized k–order n–dimensional system and a method for computing the transfer function using the DFT. Chapter 7 comprises the conclusions. Chapter 8 presents future work that can be done.

CHAPTER 2

BACKGROUND

A 2D discrete signal can be modeled as a function of two independent variables $x(i_1, i_2)$, defined for all integer values of i_1 and i_2 [25].

A 2D digital system is defined as an operator, T, that transforms an input $x(i_1, i_2)$ to an output $y(i_1, i_2)$.

$$y(i_1, i_2) = T(x(i_1, i_2))$$

For linearity,

$$T[ax(i_1, i_2) + by(i_1, i_2)] = aT[x(i_1, i_2)] + bT[y(i_1, i_2)]$$

$\forall$ a, b arbitrary constants.

The 2D z-transform of a function $x(n_1, n_2)$ is defined as:

$$X(z_1, z_2) = \sum_{i_1=-\infty}^{\infty} \sum_{i_2=-\infty}^{\infty} x(i_1, i_2) z_1^{-i_1} z_2^{-i_2}$$

where, z_1 and z_2 are complex variables.

For example a 2D system is described by the following difference equation [25]:

$$\sum_{(k_1,k_2)} \sum_{\in R_a} a(k_1, k_2) y(n_1 - k_1, n_2 - k_2) = \sum_{(k_1,k_2)} \sum_{\in R_b} b(k_1, k_2) u(n_1 - k_1, n_2 - k_2)$$

where, $u(n_1, n_2)$, $y(n_1, n_2)$ are the input and output arrays and $a(k_1, k_2)$, $b(k_1, k_2)$ are 2D finite sequences. Applying the 2D z-transform on both sides of the above equation yields,

$$H(z_1, z_2) = \frac{Y(z_1, z_2)}{U(z_1, z_2)} = \frac{\sum_{(n_1,n_2)} \sum_{\in R_b} b(n_1, n_2) z_1^{-n_1} z_2^{-n_2}}{\sum_{(n_1,n_2)} \sum_{\in R_a} a(n_1, n_2) z_1^{-n_1} z_2^{-n_2}}$$

The above numerator- denominator description $H(z_1, z_2)$ is a 2D transfer function.

2.1 2D Classical State Space models

In system theory two approaches are used for the analysis and design: the transfer function and the state space. The definition of the 2D transfer function was given and an example from the literature was presented in the previous section. It is noted that the reason for presenting the 2D case is only due to simplicity.

There are the three classical 2D state space/model descriptions:

- Givone–Roesser
- Attasi
- Fornasini–Marchesini

2.1.1 Givone–Roesser 2D model

In 1972, Givone–Roesser introduced the first spatial state space system for linear iterative circuits. Iterative circuits are a combination of individual cells [15]. The updating state equation and the output equations are:

$$\begin{aligned} \begin{bmatrix} \mathbf{x}^h_{n_1}(i_1+1,i_2) \\ \mathbf{x}^v_{n_2}(i_1,i_2+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}^h_{n_1}(i_1,i_2) \\ \mathbf{x}^v_{n_2}(i_1,i_2) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} u(i,j) \\ y(i,j) &= \begin{bmatrix} \mathbf{C}'_1 & \mathbf{C}'_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}^h_{n_1}(i_1,i_2) \\ \mathbf{x}^v_{n_2}(i_1,i_2) \end{bmatrix} + du(i_1,i_2) \end{aligned} \tag{1}$$

where,

$x^h(i,j) \in R^{n1}$.. Horizontal State Vector

$x^v(i,j) \in R^{n2}$.. Vertical State Vector

$y\ (i,j) \in R^p$.. Output Vector

$u\ (i,j) \in R^m$.. Input Vector

In a compact form the above state space model (1) can be written as:

$$\begin{aligned} \dot{\mathbf{x}}(i_1, i_2) &= \mathbf{A}\mathbf{x}(i_1, i_2) + \mathbf{B}u(i_1, i_2) \\ y(i, j) &= \mathbf{C}'\mathbf{x}(i_1, i_2) + du(i_1, i_2) \end{aligned} \tag{2}$$

where,

$$\dot{\mathbf{x}}(i_1, i_2) = \begin{bmatrix} \mathbf{x}_h(i_1 + 1, i_2) \\ \mathbf{x}_v(i_1, i_2 + 1) \end{bmatrix} \in R^{n1+n2}$$

and

$$\mathbf{x}(i_1, i_2) = \begin{bmatrix} \mathbf{x}_h(i_1, i_2) \\ \mathbf{x}_v(i_1, i_2) \end{bmatrix} \in R^{n1+n2}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix}$$

$\mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_{21}, \mathbf{A}_{22}, \mathbf{B}_1, \mathbf{B}_2, \mathbf{C}_1, \mathbf{C}_2$, are real matrices of appropriate dimensions and d is a scalar.

Applying the 2D z-transform on (2), with zero initial conditions, the corresponding 2D transfer function takes the following form:

$$H_{gr}(z_1, z_2) = \mathbf{C}'[\mathcal{Z} - \mathbf{A}]^{-1}\mathbf{B} + d \tag{3}$$

where, $\mathcal{Z} = z_1\mathbf{I}_n \oplus z_2\mathbf{I}_n$, with $\oplus$ denoting the direct sum.

2.1.2 Attasi 2D model

In 1973, S. Attasi proposed the following 2D model [5]:

$$\begin{aligned} x(i_1 + 1, i_2 + 1) &= \mathbf{A}_1 x(i_1 + 1, j) + \mathbf{A}_2 x(i_1, i_2 + 1) + \mathbf{A}_0 x(i_1, i_2) + \mathbf{B}u(i_1, i_2) \\ y(i, j) &= \mathbf{C}'\mathbf{x}(i_1, i_2) \end{aligned} \tag{4}$$

Applying the 2D z-transform the following 2D transfer function is derived,

$$H_a(z_1, z_2) = \mathbf{C}'[z_1 z_2 \mathcal{I} - z_1 \mathbf{A}_1 - z_2 \mathbf{A}_2 - \mathbf{A}_0]^{-1} \mathbf{B} \tag{5}$$

2.1.3 Fornasini-Marchesini 2D models

In 1976 and 1978, Fornasini–Marchesini proposed the following 2D models:

Fornasini–Marchesini First 2D state space model [11]:

$$\begin{aligned} x(i_1+1, i_2+1) &= \mathbf{A}_0 x(i_1, i_2) + \mathbf{A}_1 x(i_1+1, i_2) + \mathbf{A}_2 x(i_1, i_2+1) + \mathbf{B} u(i_1, i_2) \\ y(i_1, i_2) &= \mathbf{C}' \mathbf{x}(i_1, i_2) \end{aligned} \tag{6}$$

Fornasini–Marchesini First 2D transfer function:

$$H_{fm1}(z_1, z_2) = \mathbf{c}'[z_1 z_2 \mathcal{I} - z_1 \mathbf{A}_1 - z_2 \mathbf{A}_2 - \mathbf{A}_0]^{-1} \mathbf{B} \tag{7}$$

Fornasini–Marchesini Second 2D state space model [12]:

$$\begin{aligned} x(i_1+1, i_2+1) &= \mathbf{A}_1 x(i_1+1, i_2) + \mathbf{A}_2 x(i_1, i_2+1) + \mathbf{B}_1 u(i_1+1, i_2) + \mathbf{B}_2 u(i_1, i_2+1) \\ y(i, j) &= \mathbf{C}' \mathbf{x}(i_1, i_2) \end{aligned} \tag{8}$$

Fornasini–Marchesini Second 2D transfer function:

$$H_{fm1}(z_1, z_2) = \mathbf{c}'[z_1 z_2 \mathcal{I} - z_1 \mathbf{A}_1 - z_2 \mathbf{A}_2]^{-1} \mathbf{B}_1 z_1 + \mathbf{B}_2 z_2 \tag{9}$$

It is noted that the Fornasini–Marchesini models are based on the algebraic viewpoint of Nerode equivalence [7].

In the following section the mathematical representation of the new model is analyzed by defining the properties of the model. This model is of second–order and two dimensions.

CHAPTER 3

SECOND-ORDER TWO-DIMENSIONAL SYSTEM

A second–order two–dimensional (SO2D) system has the following structure [3]:

$$\begin{aligned} x(i_1+2, i_2+2) &= \mathbf{A}_0 x(i_1+1, i_2+1) + \mathbf{A}_1 x(i_1+1, i_2) + \mathbf{A}_2 x(i_1, i_2+1) \\ &\quad + \mathbf{B}_1 u(i_1+1, i_2) + \mathbf{B}_2 u(i_1, i_2+1) \\ y(i_1, i_2) &= \mathbf{C}\, x(i_1, i_2) \end{aligned} \tag{10}$$

where, $x(i_1, i_2) \in \mathcal{R}^\lambda$, $u(i_1, i_2) \in \mathcal{R}^m$, $y(i_1, i_2) \in \mathcal{R}^p$, i_1, i_2 are integer–valued vertical and horizontal coordinates respectively, $x(i_1, i_2)$ is the local vector at (i_1, i_2), $u(i_1, i_2)$ is the input vector and $y(i_1, i_2)$ is the output vector. $\mathbf{A}_k$, for $k = 0, 1, 2$ and $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{C}$, are real matrices of appropriate dimensions denoting the characteristics of the SO2D system that can also be represented by a 2D transfer function, as in the regular 2D systems [12]. It is noted that this particular SO2D system (10) is an extension of the regular 2D Fornasini–Marchesini model [12] to cover systems of second–order. The "state" circuit for the SO2D system (10) is depicted in Figure 1. For more 2D second–order structures the reader can refer to [3].

Applying the 2D z_i, $i = 1, 2$ transform to system (10), with zero initial conditions, the transfer function is found to be:

$$\mathbf{T}(z_1, z_2) = \mathbf{C}\, [\mathbf{I} z_1^2 z_2^2 - \mathbf{A}_0 z_1 z_2 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2]^{-1} \cdot [\mathbf{B}_1 z_1 + \mathbf{B}_2 z_2] \tag{11}$$

3.1 2O2D Background

To help the reader consider the SO2D model (10) with some numerical constant coefficients, for example:

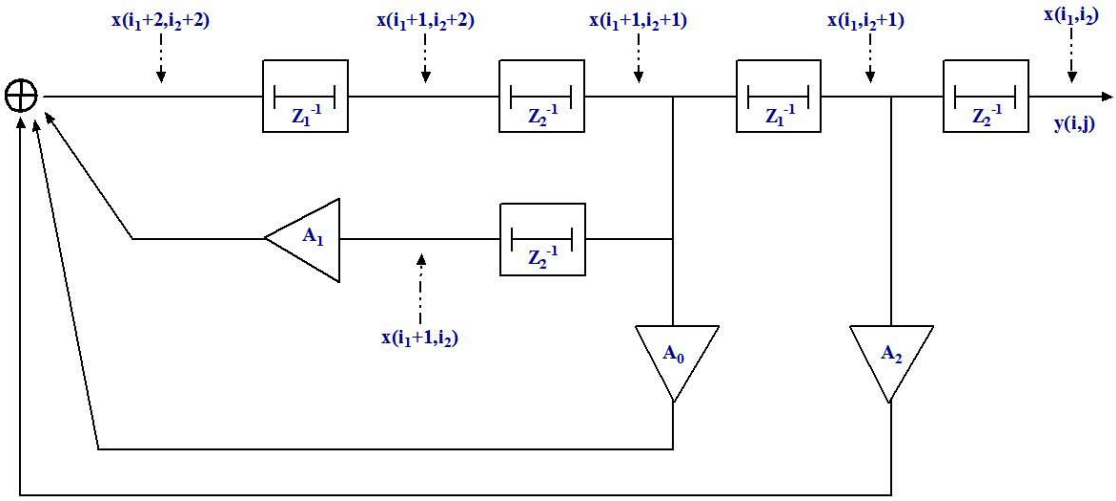

Figure 1: 2O2D state circuit

$$
\begin{aligned}
x(i_1+2, i_2+2) &= x(i_1+1, i_2+1) + 2x(i_1+1, i_2) + 3x(i_1, i_2+1) \\
&\quad + u(i_1+1, i_2) + 2u(i_1, i_2+1) \qquad (12) \\
y(i_1, i_2) &= x(i_1, i_2)
\end{aligned}
$$

It's transfer function becomes

$$T(z_1, z_2) = \frac{z_1 + 2z_2}{z_1^2 z_2^2 - z_1 z_2 - 2z_1 + 3z_2}.$$

In comparison, with the "equivalent" classical 2D Fornasini–Marchesini model [12],

$$
\begin{aligned}
x(i_1+1, i_2+1) &= 2x(i_1+1, i_2) + 3x(i_1, i_2+1) + u(i_1+1, i_2) + 2u(i_1, i_2+1) \\
y(i_1, i_2) &= x(i_1, i_2) \qquad (13)
\end{aligned}
$$

The transfer function,

$$T(z_1, z_2) = \mathbf{C}\left[\mathbf{I} z_1 z_2 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2\right]^{-1} \cdot (\mathbf{B}_1 z_1 + \mathbf{B}_2 z_2) \qquad (14)$$

or

$$T(z_1, z_2) = \frac{z_1 + 2z_2}{z_1 z_2 - 2z_1 + 3z_2}. \qquad (15)$$

The Figure 2 depicts the relation among the "state" and any point (i_1, i_2). Also the "state" at the neighborhood points $(i_1 - 1, i_2)$, $(i_1 - 1, i_2 - 1)$ and $(i_1, i_2 - 1)$ and the values of the boundary conditions $x(i_1, 0)$ and $x(0, i_2)$.

It is clear that the value of a "state" at any point depends on the entire neighbor "state" values of all the neighbor points. The boundary conditions too are simple. If for example we assume zero inputs, then the "state" at various $x(.,.)$ points are given by:

$$x(2,2) = A_0 x(1,1) + A_1 x(1,0) + A_2 x(0,1)$$

$$x(2,3) = A_0 x(1,2) + A_1 x(1,1) + A_2 x(0,2)$$

$$x(2,4) = A_0x(1,3) + A_1x(1,2) + A_2x(0,3)$$

$$x(2,5) = A_0x(1,4) + A_1x(1,3) + A_2x(0,4)$$

$$x(3,2) = A_0x(2,1) + A_1x(2,0) + A_2x(1,1)$$

$$x(3,3) = A_0x(2,2) + A_1x(2,1) + A_2x(1,2)$$

$$x(3,4) = A_0x(2,3) + A_1x(2,2) + A_2x(1,3)$$

$$x(3,5) = A_0x(2,4) + A_1x(2,3) + A_2x(1,4)$$

$$x(4,2) = A_0x(3,1) + A_1x(3,0) + A_2x(2,1)$$

$$x(4,3) = A_0x(3,2) + A_1x(3,1) + A_2x(2,2)$$

$$x(4,4) = A_0x(3,3) + A_1x(3,2) + A_2x(2,3)$$

$$x(4,5) = A_0x(3,4) + A_1x(3,3) + A_2x(2,4)$$

$$x(5,2) = A_0x(4,1) + A_1x(4,0) + A_2x(3,1)$$

$$x(5,3) = A_0x(4,2) + A_1x(4,1) + A_2x(3,2)$$

$$x(5,4) = A_0x(4,3) + A_1x(4,2) + A_2x(3,3)$$

$$x(5,5) = A_0x(4,4) + A_1x(4,3) + A_2x(3,4)$$

The Figure 2 clearly shows the above analyis.

In the following section an interpolative approach is developed for determining the transfer function $\mathbf{T}(z_1, z_2)$, given the matrices $\mathbf{A}_k$, $k = 0, 1, 2$ and $\mathbf{B}_1,\ \mathbf{B}_2,\ \mathbf{C}$ using the 2D DFT. For the sake of completeness a brief description of the 2D DFT follows.

3.2 2D Discrete Fourier Transform

Consider the finite sequences $X(k_1, k_2)$ and $\tilde{X}(r_1, r_2)$, $k_i, r_i = 0, \cdots, M_i, \quad \forall\ i = 1, 2$. In order for the sequences $X(k_1, k_2)$ and $\tilde{X}(r_1, r_2)$, to constitute a 2D DFT pair the following relations should hold [10]:

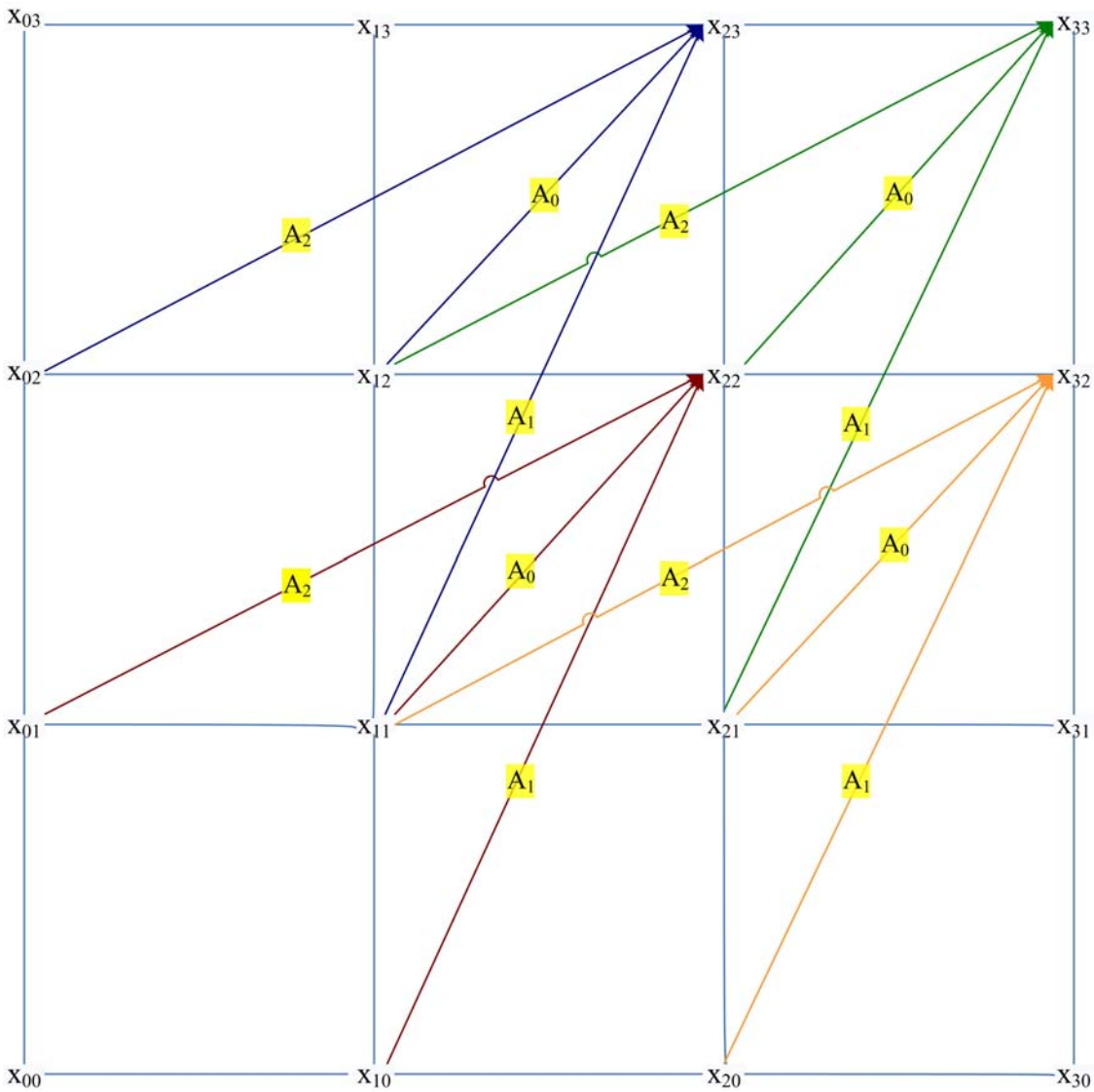

Figure 2: 2O2D time diagram

$$\tilde{X}(r_1, r_2) = \sum_{k_1=0}^{M_1} \sum_{k_2=0}^{M_2} X(k_1, k_2) W_1^{-k_1 r_1} W_2^{-k_2 r_2} \tag{16}$$

$$X(k_1, k_2) = \frac{1}{R} \sum_{r_1=0}^{M_1} \sum_{r_2=0}^{M_2} \tilde{X}(r_1, r_2) W_1^{k_1 r_1} W_2^{k_2 r_2} \tag{17}$$

where,

$$R = (M_1 + 1)(M_2 + 1) \tag{18}$$

$$W_i = e^{(2\pi j)/(M_i+1)},\ i = 1, 2 \tag{19}$$

X, $\tilde{X}$ are discrete argument matrix valued functions, with dimensions $p \times m$.

In the following section an interpolative approach is developed for determining the transfer function $\mathbf{T}(s)$, given the matrices $\mathbf{A}_i, i = 0, 1, 2$ and $\mathbf{B}_1,\ \mathbf{B}_2,\ \mathbf{C}$, using the 2D DFT.

3.3 Algorithm

The transfer function of a SO2D system (10) is, +

$$\mathbf{T}(z_1, z_2) = \frac{\mathbf{N}(z_1, z_2)}{d(z_1, z_2)} \tag{20}$$

where,

$$\mathbf{N}(z_1, z_2) = \mathbf{C}\ \mathtt{adj}\ [\mathbf{I} z_1^2 z_2^2 - \mathbf{A}_0 z_1 z_2 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2] \cdot (\mathbf{B}_1 z_1 + \mathbf{B}_2 z_2) \tag{21}$$

$$d(z_1, z_2) = \mathtt{det}\ [\mathbf{I} z_1^2 z_2^2 - \mathbf{A}_0 z_1 z_2 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2] \tag{22}$$

Equations (21) and (22) can be written in polynomial form as follows:

$$\mathbf{N}(z_1, z_2) = \sum_{\lambda_1=0}^{n_{max}^P} \sum_{\lambda_2=0}^{n_{max}^P} \mathbf{P}_{\lambda_1,\lambda_2} z_1^{\lambda_1} z_2^{\lambda_2} \tag{23}$$

with, $n_{max}^P := max((2\lambda_1 - 1), (2\lambda_2 - 1))$. The numerator coefficients $\mathbf{P}_{\lambda_1,\lambda_2}$ are matrices with dimensions $(p \times m)$.

$$d(z_1, z_2) = \sum_{\lambda_1=0}^{n^q_{max}} \sum_{\lambda_2=0}^{n^q_{max}} q_{\lambda_1,\lambda_2} z_1^{\lambda_1} z_2^{\lambda_2} \tag{24}$$

where, $n^q_{max} := max(2\lambda_1, 2\lambda_2)$. The denominator coefficients q_{λ_1,λ_2} are scalars.

The numerator polynomial matrix $\mathbf{N}(z_1, z_2)$ and the denominator polynomial $d(z_1, z_2)$ can be numerically computed at $R = (r+1)^2$, points, equally spaced on the unit 2D disc. The R points are chosen as $(z_1, z_2) = [v(i_1), v(i_2)],\ i_1, i_2 = 0, \cdots, r$, with $r = 2\lambda$, according to definition as:

$$v_1(i) = v_2(i) = W^{-i},\ \ \forall\ \ i = 0, \ldots, r. \tag{25}$$

where,

$$W_i = e^{(2\pi j)/(r+1)},\ \ i = 1, 2 \tag{26}$$

The values of the transfer function (20) at the R points are the corresponding 2D DFT coefficients.

3.3.1 Denominator Polynomial

To evaluate the denominator coefficients q_{λ_1,λ_2}, define,

$$a_{i_1,i_2} = \texttt{det}\ [\mathbf{I} v_1^2(i_1) v_2^2(i_2) - \mathbf{A}_0 v_1(i_1) v_2(i_2) - \mathbf{A}_1 v_1(i_1) - \mathbf{A}_2 v_2(i_2)] \tag{27}$$

Therefore using equations (24) and (27), a_{i_1,i_2} can be defined as,

$$a_{i_1,i_2} = d[v_1(i_1), v_2(i_2)] \tag{28}$$

Provided that at least one of $a_{i_1,i_2} \neq 0$.

Equations (24), (25) and (28) yield

$$a_{i_1,i_2} = \sum_{\lambda_1=0}^{r} \sum_{\lambda_2=0}^{r} q_{\lambda_1,\lambda_2} W^{-(i_1\lambda_1 + i_2\lambda_2)} \tag{29}$$

In the above equation (29) it is obvious that $[a_{i_1,i_2}]$ and $[q_{\lambda_1,\lambda_2}]$ form a 2D DFT pair. Therefore the coefficients $[q_{\lambda_1,\lambda_2}]$ can be computed using the inverse 2D DFT, as follows:

$$q_{\lambda_1,\lambda_2} = \frac{1}{R}\sum_{i_1=0}^{r}\sum_{i_2=0}^{r} a_{i_1,i_2} W^{(i_1\lambda_1+i_2\lambda_2)} \tag{30}$$

3.3.2 Numerator Polynomial

To evaluate the numerator matrix polynomial $\mathbf{P}_{\lambda_1,\lambda_2}$, define

$$\begin{aligned}\mathbf{F}_{i_1,i_2} &= \mathbf{C}\ \texttt{adj}\ [\mathbf{I}v_1^2(i_1)v_2^2(i_2) - \mathbf{A}_0 v_1(i_1)v_2(i_2) - \mathbf{A}_1 v_1(i_1) - \mathbf{A}_2 v_2(i_2)] \\ &\quad \times[\mathbf{B}_1 v_1(i_1) + \mathbf{B}_2 v_2(i_2)] \end{aligned} \tag{31}$$

Using equations (23) and (31), $\mathbf{F}_{i_1,i_2}$ can be defined as,

$$\mathbf{F}_{i_1,i_2} = \mathbf{N}[v_1(i_1), v_2(i_2)] \tag{32}$$

Equations (23), (25) and (32) yield

$$\mathbf{F}_{i_1,i_2} = \sum_{\lambda_1=0}^{r-1}\sum_{\lambda_2=0}^{r-1} \mathbf{P}_{\lambda_1,\lambda_2} W^{-(i_1\lambda_1+i_2\lambda_2)} \tag{33}$$

In the above equation (29), $[\mathbf{F}_{i_1,i_2}]$, $[\mathbf{P}_{\lambda_1,\lambda_2}]$ form a 2D DFT pair. Therefore the coefficients $\mathbf{P}_{\lambda_1,\lambda_2}$ can be computed, using the inverse 2D DFT, as follows:

$$\mathbf{P}_{\lambda_1,\lambda_2} = \frac{1}{R}\sum_{i_1=0}^{r-1}\sum_{i_2=0}^{r-1} \mathbf{F}_{i_1,i_2} W^{(i_1\lambda_1+i_2\lambda_2)} \tag{34}$$

Three salient examples, simple yet illustrative of the theoretical concepts presented in this work, follow below:

3.4 Numerical Examples

3.4.1 Single–Input Single–Output

Consider the following single–input single–output 2DSO system:

$$\begin{aligned} x(i_1+2, i_2+2) &= \mathbf{A}_0 x(i_1+1, i_2+1) + \mathbf{A}_1 x(i_1+1, i_2) + \mathbf{A}_2 x(i_1, i_2+1) \\ &\quad + \mathbf{B}_1 u(i_1+1, i_2) + \mathbf{B}_2 u(i_1, i_2+1) \\ y(i_1, i_2) &= \mathbf{C}\ x(i_1, i_2) \end{aligned} \tag{35}$$

where,

$$\mathbf{A}_0 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \ \mathbf{A}_1 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \ \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Since $\lambda = 2$, the $r = 2 \cdot \lambda = 4$. Therefore $R = (r+1)^2 = 25$. The direct application of the proposed algorithm yields:

Using (29), the $a_{i_1 i_2}$ coefficients are,

$$\begin{aligned}
a_{00} &= -6.0000 + 0.0000j, & a_{01} &= 3.0451 + 2.4899j \\
a_{02} &= -2.5451 + 0.2245j & a_{03} &= -2.5451 - 0.2245j \\
a_{04} &= 3.0451 - 2.4899j & a_{10} &= -0.3090 + 4.7553j \\
a_{11} &= 1.9271 + 3.3022j & a_{12} &= 0.1180 - 2.7144j \\
a_{13} &= 3.3090 + 1.4001j & a_{14} &= -1.0000 - 3.8042j \\
a_{20} &= 0.8090 + 2.9389j & a_{21} &= 2.1910 - 4.3920j \\
a_{22} &= -1.4271 - 3.2164j & a_{23} &= -1.0000 - 2.3511j \\
a_{24} &= -2.1180 + 2.2654j & a_{30} &= 0.8090 - 2.9389j \\
a_{31} &= -2.1180 - 2.2654j & a_{32} &= -1.0000 + 2.3511j \\
a_{33} &= -1.4271 + 3.2164j & a_{34} &= 2.1910 + 4.3920j \\
a_{40} &= -0.3090 - 4.7553j & a_{41} &= -1.0000 + 3.8042j \\
a_{42} &= 3.3090 - 1.4001j & a_{43} &= 0.1180 + 2.7144j \\
a_{44} &= 1.9271 - 3.3022j
\end{aligned}$$

The Figure 3 shows the plot of a_{i_1,i_2} (29), using $MATLAB^{TM}$, given in Appendix-A.

Using (33), the $F_{i_1 i_2}$ coefficients are,

$$
\begin{aligned}
F_{00} &= 2.0000 + 0.0000j & F_{01} &= 0.6180 - 0.7265j \\
F_{02} &= -1.6180 - 3.0777j & F_{03} &= -1.6180 + 3.0777j \\
F_{04} &= 0.6180 + 0.7265j & F_{10} &= -0.5000 - 1.5388j \\
F_{11} &= 0.1910 - 1.7634j & F_{12} &= -3.4271 + 1.7634j \\
F_{13} &= 1.7361 + 1.5388j & F_{14} &= 2.0000 + 0.0000j \\
F_{20} &= -0.5000 + 0.3633j & F_{21} &= -2.7361 - 0.3633j \\
F_{22} &= 1.3090 + 2.8532j & F_{23} &= 2.0000 + 0.0000j \\
F_{24} &= -0.0729 - 2.8532j & F_{30} &= -0.5000 - 0.3633j \\
F_{31} &= -0.0729 + 2.8532j & F_{32} &= 2.0000 - 0.0000j \\
F_{33} &= 1.3090 - 2.8532j & F_{34} &= -2.7361 + 0.3633j \\
F_{40} &= -0.5000 + 1.5388j & F_{41} &= 2.0000 - 0.0000j \\
F_{42} &= 1.7361 - 1.5388j & F_{43} &= 3.4271 - 1.7634j \\
F_{44} &= 0.1910 + 1.7634j
\end{aligned}
$$

The Figure 3 shows the plot of $\mathbf{F}_{i_1,i_2}$ (33), using the software package $MATLAB^{TM}$, given in Appendix-A.

Using (30), the denominator coefficients are,

$$
\begin{bmatrix}
q_{00} & q_{01} & q_{02} & q_{03} & q_{04} \\
q_{10} & q_{11} & q_{12} & q_{13} & q_{14} \\
q_{20} & q_{21} & q_{22} & q_{23} & q_{24} \\
q_{30} & q_{31} & q_{32} & q_{33} & q_{34} \\
q_{40} & q_{41} & q_{42} & q_{43} & q_{44}
\end{bmatrix}
=
\begin{bmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & -2 & -2 & 0 & 0 \\
-1 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Using (34), the numerator matrix polynomials are,

$$
\begin{bmatrix}
P_{00} & P_{01} & P_{02} & P_{03} & P_{04} \\
P_{10} & P_{11} & P_{12} & P_{13} & P_{14} \\
P_{20} & P_{21} & P_{22} & P_{23} & P_{24} \\
P_{30} & P_{31} & P_{32} & P_{33} & P_{34} \\
P_{40} & P_{41} & P_{42} & P_{43} & P_{44}
\end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

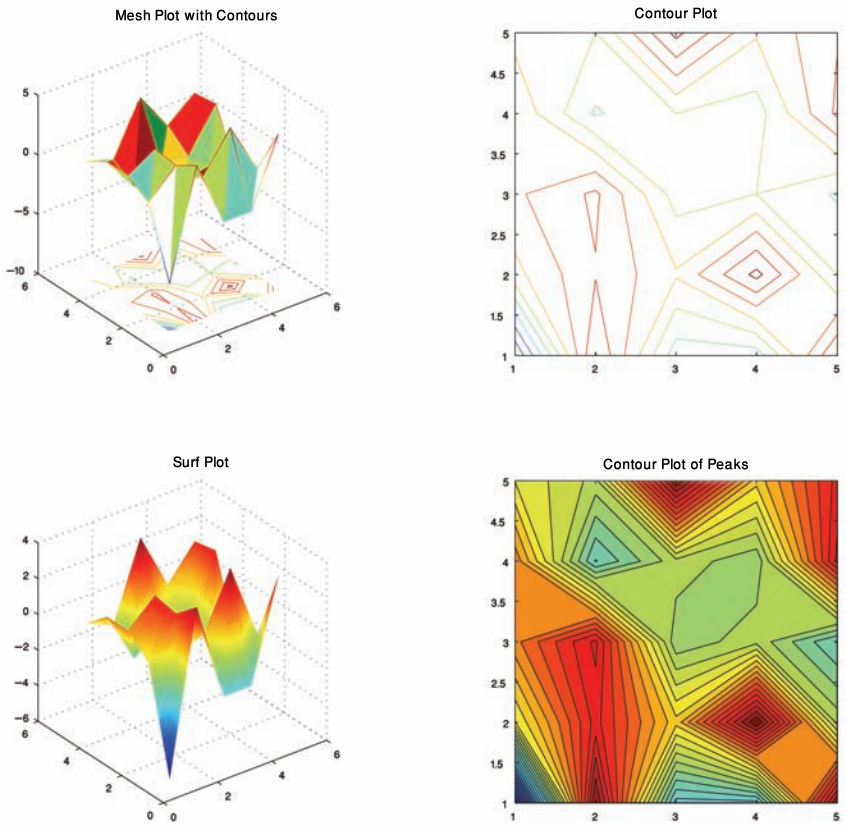

Figure 3: Plot of a_{i_1,i_2} (29)

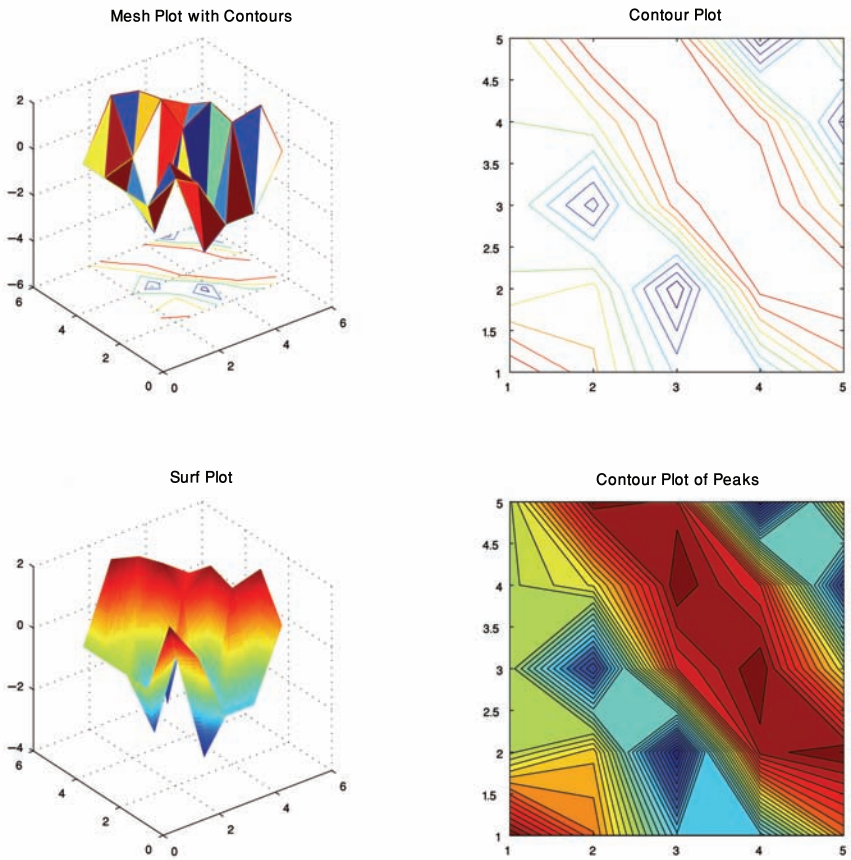

Figure 4: Plot of $\mathbf{F}_{i_1,i_2}$ (33)

Once the denominator and the adjoint matrix coefficients have been computed, the transfer function $T(z_1, z_2)$ is determined as,

$$T(z_1, z_2) = \frac{P_{23} z_1^2 z_2^3 + P_{12} z_1 z_2^2 + P_{11} z_1 z_2}{q_{44} z_1^4 z_2^4 + q_{33} z_1^3 z_2^3 + q_{23} z_1^2 z_2^3 + q_{21} z_1^2 z_2 + q_{20} z_1^2 + q_{12} z_1 z_2^2 + q_{11} z_1 z_2 + q_{02} z_2^2}$$

or

$$T(z_1, z_2) = \frac{z_1^2 z_2^3 - z_1 z_2^2 + z_1 z_2}{z_1^4 z_2^4 - z_1^3 z_2^3 - z_1^2 z_2^3 + z_1^2 z_2 - z_1^2 - 2 z_1 z_2^2 - 2 z_1 z_2 - z_2^2} \tag{36}$$

Figure 5 shows the mesh, contour, surface and z plots of the above 2D transfer function (36). The $MATLAB^{TM}$ code is given in the Appendix B.

The correctness of the above result (36) can easily be verified using (11), with $k = n = 2$,

$$T(z_1, z_2) = \mathbf{C}\,[\mathbf{I} z_1^2 z_2^2 - \mathbf{A}_0 z_1 z_2 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2]^{-1} \cdot (\mathbf{B}_1 z_1 + \mathbf{B}_2 z_2)$$

3.4.2 Multiple–Input Single–Output

Consider the following two–input single–output 2DSO system:

$$\begin{aligned} x(i_1+2, i_2+2) &= \mathbf{A}_0 x(i_1+1, i_2+1) + \mathbf{A}_1 x(i_1+1, i_2) + \mathbf{A}_2 x(i_1, i_2+1) \\ &\quad + \mathbf{B}_1 u(i_1+1, i_2) + \mathbf{B}_2 u(i_1, i_2+1) \\ y(i_1, i_2) &= \mathbf{C}\, x(i_1, i_2) \end{aligned} \tag{37}$$

where,

$$\begin{aligned} \mathbf{A}_0 &= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \ \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\ \mathbf{B}_1 &= \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \\ \mathbf{C} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned}$$

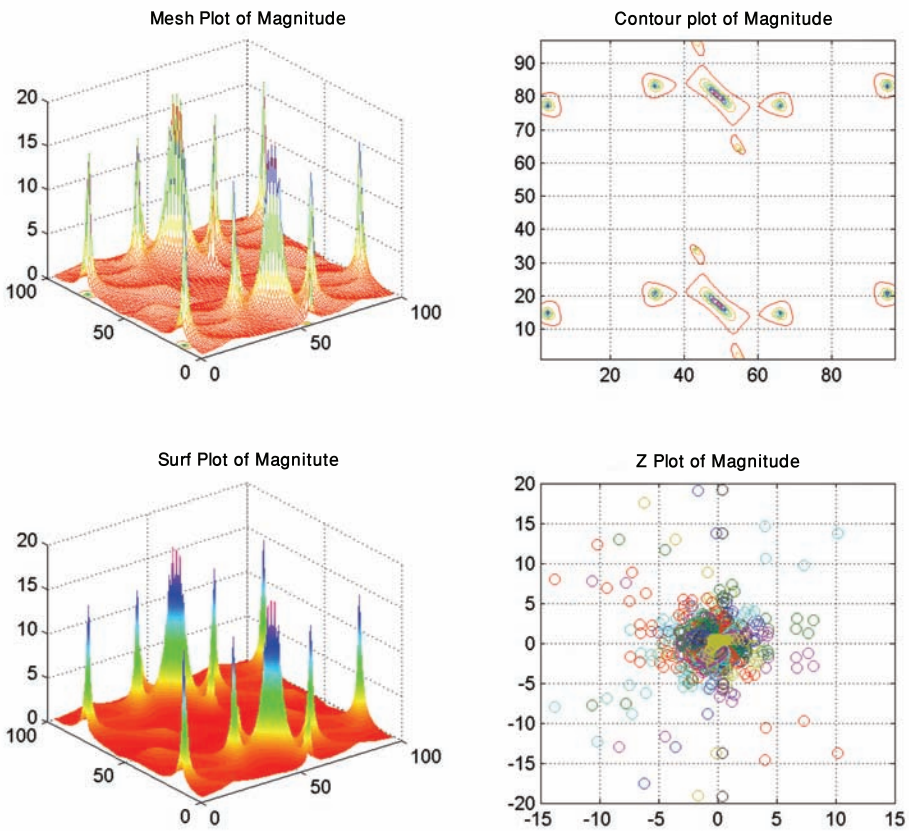

Figure 5: Example 3.4.1

Since $\lambda = 2$, the $r = 2 \cdot \lambda = 4$. Therefore $R = (r+1)^2 = 25$. The direct application of the proposed algorithm yields:

The coefficients (29) $a_{i_1 i_2}$ $\forall$ $i_1, i_2 = 0, ..., 4$ are the same as in the single-input single-output Example 3.4.1

Using (33), the $F_{i_1 i_2}$ coefficients are,

$$
\begin{aligned}
\mathbf{F}_{00} &= \begin{bmatrix} -19.0000 & 2.0000 \end{bmatrix} \\
\mathbf{F}_{01} &= \begin{bmatrix} -9.7533 + 18.1558j & -7.4721 - 0.4490j \end{bmatrix} \\
\mathbf{F}_{02} &= \begin{bmatrix} 9.2533 + 4.6493j & 1.4721 + 4.9798j \end{bmatrix} \\
\mathbf{F}_{03} &= \begin{bmatrix} 9.2533 - 4.6493j & 1.4721 - 4.9798j \end{bmatrix} \\
\mathbf{F}_{04} &= \begin{bmatrix} -9.7533 - 18.1558j & -7.4721 + 0.4490j \end{bmatrix} \\
\mathbf{F}_{10} &= \begin{bmatrix} 2.1180 + 13.1230j & -0.5000 + 0.8123j \end{bmatrix} \\
\mathbf{F}_{11} &= \begin{bmatrix} 22.6074 + 2.9389j & 3.8090 + 4.1145j \end{bmatrix} \\
\mathbf{F}_{12} &= \begin{bmatrix} 1.9549 - 12.0005j & 3.8090 - 0.5878j \end{bmatrix} \\
\mathbf{F}_{13} &= \begin{bmatrix} -7.6180 - 3.3552j & -0.5000 - 1.5388j \end{bmatrix} \\
\mathbf{F}_{14} &= \begin{bmatrix} -2.8820 + 11.0494j & 1.4721 + 3.0777j \end{bmatrix} \\
\mathbf{F}_{20} &= \begin{bmatrix} -0.1180 - 6.3471j & -0.5000 - 3.4410j \end{bmatrix} \\
\mathbf{F}_{21} &= \begin{bmatrix} -5.3820 - 7.3309j & -0.5000 + 0.3633j \end{bmatrix} \\
\mathbf{F}_{22} &= \begin{bmatrix} -3.1074 - 4.7553j & 2.6910 - 6.6574j \end{bmatrix} \\
\mathbf{F}_{23} &= \begin{bmatrix} -5.1180 + 5.5146j & -7.4721 - 0.7265j \end{bmatrix} \\
\mathbf{F}_{24} &= \begin{bmatrix} 7.5451 - 6.1024j & 2.6910 + 0.9511j \end{bmatrix} \\
\mathbf{F}_{30} &= \begin{bmatrix} -0.1180 + 6.3471j & -0.5000 + 3.4410j \end{bmatrix} \\
\mathbf{F}_{31} &= \begin{bmatrix} 7.5451 + 6.1024j & 2.6910 - 0.9511j \end{bmatrix} \\
\mathbf{F}_{32} &= \begin{bmatrix} -5.1180 - 5.5146j & -7.4721 + 0.7265j \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}_{33} &= \begin{bmatrix} -3.1074 + 4.7553j & 2.6910 + 6.6574j \end{bmatrix} \\
\mathbf{F}_{34} &= \begin{bmatrix} -5.3820 + 7.3309j & -0.5000 - 0.3633j \end{bmatrix} \\
\mathbf{F}_{40} &= \begin{bmatrix} 2.1180 - 13.1230j & -0.5000 - 0.8123j \end{bmatrix} \\
\mathbf{F}_{41} &= \begin{bmatrix} -2.8820 - 11.0494j & 1.4721 - 3.0777j \end{bmatrix} \\
\mathbf{F}_{42} &= \begin{bmatrix} -7.6180 + 3.3552j & -0.5000 + 1.5388j \end{bmatrix} \\
\mathbf{F}_{43} &= \begin{bmatrix} 1.9549 + 12.0005j & 3.8090 + 0.5878j \end{bmatrix} \\
\mathbf{F}_{44} &= \begin{bmatrix} 22.6074 - 2.9389j & 3.8090 - 4.1145j \end{bmatrix}
\end{aligned}
$$

Using (30), the denominator coefficients are,

$$
\begin{bmatrix} q_{00} & q_{01} & q_{02} & q_{03} & q_{04} \\ q_{10} & q_{11} & q_{12} & q_{13} & q_{14} \\ q_{20} & q_{21} & q_{22} & q_{23} & q_{24} \\ q_{30} & q_{31} & q_{32} & q_{33} & q_{34} \\ q_{40} & q_{41} & q_{42} & q_{43} & q_{44} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

Using (34), the numerator matrix polynomials are,

$$
\begin{bmatrix} P_{00} & P_{01} & \mathbf{P}_{02} & P_{03} & P_{04} \\ P_{10} & \mathbf{P}_{11} & \mathbf{P}_{12} & P_{13} & P_{14} \\ \mathbf{P}_{20} & \mathbf{P}_{21} & P_{22} & \mathbf{P}_{23} & P_{24} \\ P_{30} & P_{31} & \mathbf{P}_{32} & P_{33} & P_{34} \\ P_{40} & P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \begin{bmatrix} -3 & 0 \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} -7 & -2 \end{bmatrix} & \begin{bmatrix} 0 & 3 \end{bmatrix} & 0 & 0 \\ \begin{bmatrix} -4 & -2 \end{bmatrix} & \begin{bmatrix} -9 & 0 \end{bmatrix} & 0 & \begin{bmatrix} 3 & 1 \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} 1 & 2 \end{bmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

Once the denominator and the adjoint matrix coefficients have been computed, the transfer function $\mathbf{T}(z_1, z_2)$ is determined as,

$$
\mathbf{T}(z_1, z_2) = \frac{\mathbf{P}_{02}z_2^2 + \mathbf{P}_{11}z_1z_2 + \mathbf{P}_{12}z_1z_2^2 + \mathbf{P}_{20}z_1^2 + \mathbf{P}_{21}z_1^2z_2 + \mathbf{P}_{23}z_1^2z_2^3 + \mathbf{P}_{32}z_1^3z_2^2}{q_{02}z_2^2 + +q_{11}z_1z_2 + q_{12}z_1z_2^2 + q_{20}z_1^2 + q_{21}z_1^2z_2 + q_{23}z_1^2z_2^3 + q_{33}z_1^3z_2^3 + q_{44}z_1^4z_2^4}
$$

or

$$\mathbf{T}(z_1,z_2)=\frac{\mathbf{P}_{02}z_2^2+\mathbf{P}_{11}z_1z_2+\mathbf{P}_{12}z_1z_2^2+\mathbf{P}_{20}z_1^2+\mathbf{P}_{21}z_1^2z_2+\mathbf{P}_{23}z_1^2z_2^3+\mathbf{P}_{32}z_1^3z_2^2}{-z_2^2-2z_1z_2-2z_1z_2^2-z_1^2+z_1^2z_2-z_1^2z_2^3-z_1^3z_2^3+z_1^4z_2^4}$$

Where,

$$\begin{aligned}
\mathbf{P}_{02} &= \begin{bmatrix} -3 & 0 \end{bmatrix},\ \mathbf{P}_{11}=\begin{bmatrix} -7 & -2 \end{bmatrix}\\
\mathbf{P}_{12} &= \begin{bmatrix} 0 & 3 \end{bmatrix},\ \mathbf{P}_{20}=\begin{bmatrix} -4 & -2 \end{bmatrix}\\
\mathbf{P}_{21} &= \begin{bmatrix} -9 & 0 \end{bmatrix},\ \mathbf{P}_{23}=\begin{bmatrix} 3 & 1 \end{bmatrix}\\
\mathbf{P}_{32} &= \begin{bmatrix} 1 & 2 \end{bmatrix}
\end{aligned}$$

or

$$\mathbf{T}(z_1,z_2)=\frac{\begin{bmatrix} \alpha_1 & | & \alpha_2 \end{bmatrix}}{-z_2^2-2z_1z_2-2z_1z_2^2-z_1^2+z_1^2z_2-z_1^2z_2^3-z_1^3z_2^3+z_1^4z_2^4} \tag{38}$$

Where,

$$\begin{aligned}
\alpha_1 &= -3z_2^2-7z_1z_2-4z_1^2-9z_1^2z_2+3z_1^2z_2^3+z_1^3z_2^2\\
\alpha_2 &= -2z_1z_2+3z_1z_2^2-2z_1^2+z_1^2z_2^3+2z_1^3z_2^2
\end{aligned}$$

Figures 6 and 7 show the mesh, contour, surface and z plots of the above 2D transfer function (38). The $MATLAB^{TM}$ code is given in the Appendix B.

The correctness of the above result (38 can easily be verified using (11) with $k=n=2$,

$$T(z_1,z_2)=\mathbf{C}\left[\mathbf{I}z_1^2z_2^2-\mathbf{A}_0z_1z_2-\mathbf{A}_1z_1-\mathbf{A}_2z_2\right]^{-1}\cdot(\mathbf{B}_1z_1+\mathbf{B}_2z_2)$$

3.4.3 Multiple–Input Multiple–Output

Consider the following multivariable (two–input two–output) 2DSO system:

$$x(i_1+2,i_2+2) = \mathbf{A}_0^1x(i_1+1,i_2+1)+\mathbf{A}_1x(i_1+1,i_2)+\mathbf{A}_2x(i_1,i_2+1)$$

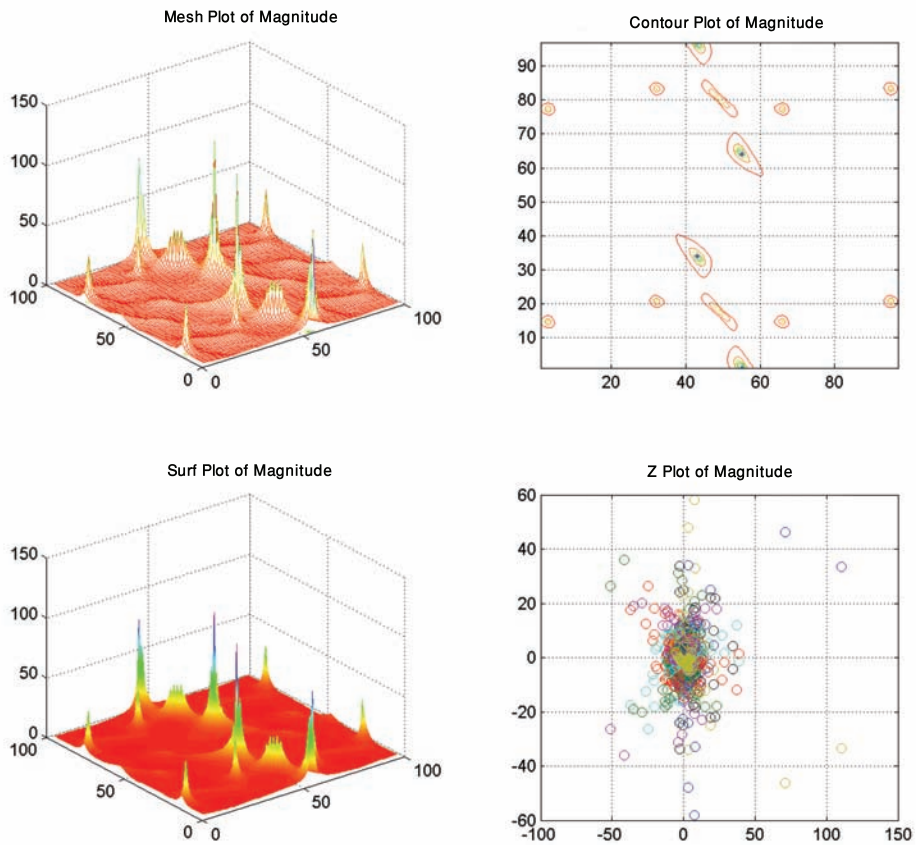

Figure 6: Example 3.4.2 - First Input

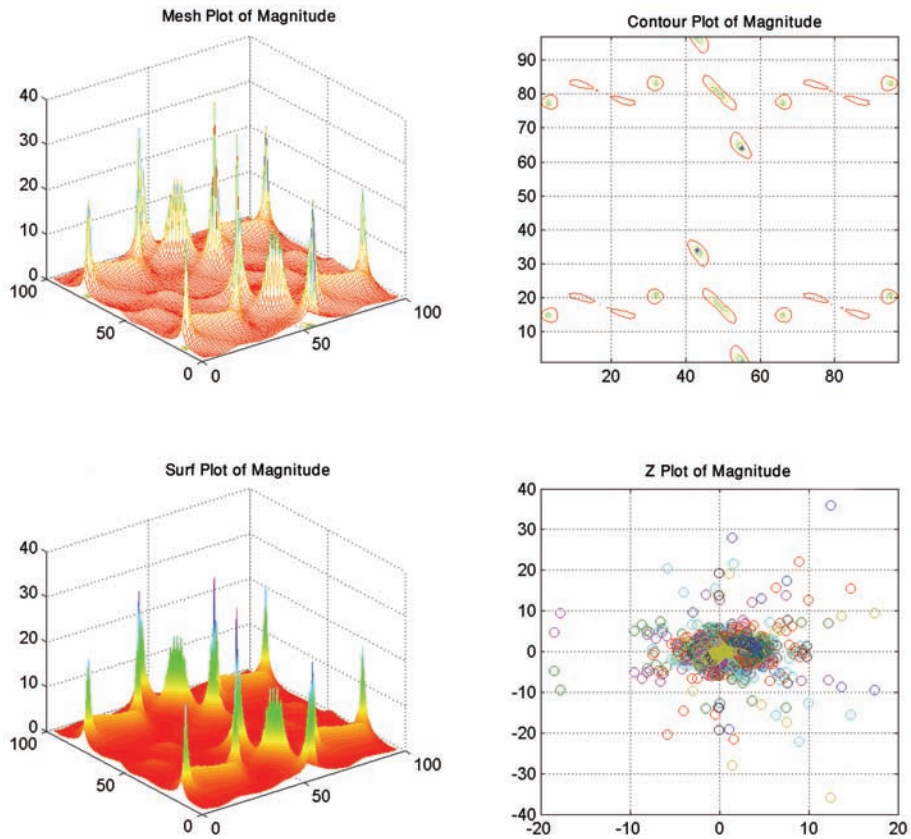

Figure 7: Example 3.4.2 - Second Input

$$
\begin{aligned}
& +\mathbf{B}_1 u(i_1+1, i_2) + \mathbf{B}_2 u(i_1, i_2+1) \\
y(i_1, i_2) &= \mathbf{C}\, x(i_1, i_2) \qquad (39)
\end{aligned}
$$

where,

$$
\begin{aligned}
\mathbf{A}_0 &= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \ \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\
\mathbf{B}_1 &= \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \\
\mathbf{C} &= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}
\end{aligned}
$$

Since $\gamma = 2$, the $r = n \times \gamma = 4$. Therefore $R = (r+1)^2 = 25$. The direct application of the proposed algorithm yields:

The coefficients (29) $a_{i_1 i_2}$ $\forall$ $i_1, i_2 = 0, ..., 4$ are the same as in the single-input single-output Example 3.4.1.

Using (33), the $F_{i_1 i_2}$ coefficients are,

$$
\begin{aligned}
\mathbf{F}_{00} &= \begin{bmatrix} 1.0000 & 4.0000 \\ -37.0000 & 8.0000 \end{bmatrix} \\
\mathbf{F}_{01} &= \begin{bmatrix} -1.7639 - 3.8042i & 0.1180 - 2.2654i \\ -21.2705 + 32.5074i & -14.8262 - 3.1634i \end{bmatrix} \\
\mathbf{F}_{02} &= \begin{bmatrix} -6.2361 - 2.3511j & -2.1180 - 2.7144j \\ 12.2705 + 6.9474j & 0.8262 + 7.2452j \end{bmatrix} \\
\mathbf{F}_{03} &= \begin{bmatrix} -6.2361 + 2.3511j & -2.1180 + 2.7144j \\ 12.2705 - 6.9474j & 0.8262 - 7.2452j \end{bmatrix} \\
\mathbf{F}_{04} &= \begin{bmatrix} -1.7639 + 3.8042j & 0.1180 + 2.2654j \\ -21.2705 - 32.5074j & -14.8262 + 3.1634j \end{bmatrix}
\end{aligned}
$$

$$\mathbf{F}_{10} = \begin{bmatrix} 5.7361 - 2.7144j & 0.8090 - 2.4899j \\ 9.9721 + 23.5317j & -0.1910 - 0.8653j \end{bmatrix}$$

$$\mathbf{F}_{11} = \begin{bmatrix} -6.2361 + 1.1756j & -1.4271 - 2.9389j \\ 38.9787 + 7.0534j & 6.1910 + 5.2901j \end{bmatrix}$$

$$\mathbf{F}_{12} = \begin{bmatrix} 5.8992 + 9.8208j & -3.9271 + 3.3022j \\ 9.8090 - 14.1801j & 3.6910 + 2.1266j \end{bmatrix}$$

$$\mathbf{F}_{13} = \begin{bmatrix} 5.7361 - 2.9919j & 2.3541 + 1.5388j \\ -9.5000 - 9.7023j & 1.3541 - 1.5388j \end{bmatrix}$$

$$\mathbf{F}_{14} = \begin{bmatrix} 1.0000 + 3.5267j & 2.1910 + 0.5878j \\ -4.7639 + 25.6255j & 5.1353 + 6.7432j \end{bmatrix}$$

$$\mathbf{F}_{20} = \begin{bmatrix} 1.2639 + 2.2654j & -0.3090 - 0.2245j \\ 1.0279 - 10.4289j & -1.3090 - 7.1064j \end{bmatrix}$$

$$\mathbf{F}_{21} = \begin{bmatrix} 1.2639 - 5.7921j & -4.3541 - 0.3633j \\ -9.5000 - 20.4540j & -5.3541 + 0.3633j \end{bmatrix}$$

$$\mathbf{F}_{22} = \begin{bmatrix} -1.7639 - 1.9021j & 1.9271 + 4.7553j \\ -7.9787 - 11.4127j & 7.3090 - 8.5595j \end{bmatrix}$$

$$\mathbf{F}_{23} = \begin{bmatrix} 1.0000 - 5.7063j & 3.3090 - 0.9511j \\ -9.2361 + 5.3228j & -11.6353 - 2.4041j \end{bmatrix}$$

$$\mathbf{F}_{24} = \begin{bmatrix} -6.3992 - 3.1307j & -0.5729 - 3.2164j \\ 8.6910 - 15.3354j & 4.8090 - 1.3143j \end{bmatrix}$$

$$\mathbf{F}_{30} = \begin{bmatrix} 1.2639 - 2.2654j & -0.3090 + 0.2245j \\ 1.0279 + 10.4289j & -1.3090 + 7.1064j \end{bmatrix}$$

$$\mathbf{F}_{31} = \begin{bmatrix} -6.3992 + 3.1307j & -0.5729 + 3.2164j \\ 8.6910 + 15.3354j & 4.8090 + 1.3143j \end{bmatrix}$$

$$\mathbf{F}_{32} = \begin{bmatrix} 1.0000 + 5.7063j & 3.3090 + 0.9511j \\ -9.2361 - 5.3228j & -11.6353 + 2.4041j \end{bmatrix}$$

$$
\begin{aligned}
\mathbf{F}_{33} &= \begin{bmatrix} -1.7639 + 1.9021j & 1.9271 - 4.7553j \\ -7.9787 + 11.4127j & 7.3090 + 8.5595j \end{bmatrix} \\
\mathbf{F}_{34} &= \begin{bmatrix} 1.2639 + 5.7921j & -4.3541 + 0.3633j \\ -9.5000 + 20.4540j & -5.3541 - 0.3633j \end{bmatrix} \\
\mathbf{F}_{40} &= \begin{bmatrix} 5.7361 + 2.7144j & 0.8090 + 2.4899j \\ 9.9721 - 23.5317j & -0.1910 + 0.8653j \end{bmatrix} \\
\mathbf{F}_{41} &= \begin{bmatrix} 1.0000 - 3.5267j & 2.1910 - 0.5878j \\ -4.7639 - 25.6255j & 5.1353 - 6.7432j \end{bmatrix} \\
\mathbf{F}_{42} &= \begin{bmatrix} 5.7361 + 2.9919j & 2.3541 - 1.5388j \\ -9.5000 + 9.7023j & 1.3541 + 1.5388j \end{bmatrix} \\
\mathbf{F}_{43} &= \begin{bmatrix} 5.8992 - 9.8208j & -3.9271 - 3.3022j \\ 9.8090 + 14.1801j & 3.6910 - 2.1266j \end{bmatrix} \\
\mathbf{F}_{44} &= \begin{bmatrix} -6.2361 - 1.1756j & -1.4271 + 2.9389j \\ 38.9787 - 7.0534j & 6.1910 - 5.2901j \end{bmatrix}
\end{aligned}
$$

Using (30), the denominator coefficients are,

$$
\begin{bmatrix} q_{00} & q_{01} & q_{02} & q_{03} & q_{04} \\ q_{10} & q_{11} & q_{12} & q_{13} & q_{14} \\ q_{20} & q_{21} & q_{22} & q_{23} & q_{24} \\ q_{30} & q_{31} & q_{32} & q_{33} & q_{34} \\ q_{40} & q_{41} & q_{42} & q_{43} & q_{44} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

Using (34), the numerator matrix polynomials are,

$$
\begin{bmatrix} P_{00} & P_{01} & \mathbf{P}_{02} & P_{03} & P_{04} \\ P_{10} & \mathbf{P}_{11} & \mathbf{P}_{12} & P_{13} & P_{14} \\ \mathbf{P}_{20} & \mathbf{P}_{21} & P_{22} & \mathbf{P}_{23} & P_{24} \\ P_{30} & P_{31} & \mathbf{P}_{32} & P_{33} & P_{34} \\ P_{40} & P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix}
$$

where the matrices $P_{00}, ..., P_{44}$ are,

$$\begin{bmatrix} 0 & 0 & \begin{bmatrix} 3 & 1 \\ -3 & 1 \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} 1 & 3 \\ -13 & -1 \end{bmatrix} & \begin{bmatrix} 0 & -1 \\ 0 & 5 \end{bmatrix} & 0 & 0 \\ \begin{bmatrix} -3 & 0 \\ -11 & -4 \end{bmatrix} & \begin{bmatrix} 3 & 0 \\ -15 & 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 & 1 \\ 6 & 3 \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} -3 & 0 \\ -1 & 4 \end{bmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Once the denominator and the adjoint matrix coefficients have been computed the transfer function, $\mathbf{T}(z_1, z_2)$, of the two–input two–output 2D2O system (39), becomes

$$\mathbf{T}(z_1, z_2) = \frac{\mathbf{P}_{02}z_2^2 + \mathbf{P}_{11}z_1z_2 + \mathbf{P}_{12}z_1z_2^2 + \mathbf{P}_{20}z_1^2 + \mathbf{P}_{21}z_1^2z_2 + \mathbf{P}_{23}z_1^2z_2^3 + \mathbf{P}_{32}z_1^3z_2^2}{q_{02}z_2^2 + q_{11}z_1z_2 + q_{12}z_1z_2^2 + q_{20}z_1^2 + q_{21}z_1^2z_2 + q_{23}z_1^2z_2^3 + q_{33}z_1^3z_2^3 + q_{44}z_1^4z_2^4}$$

or

$$\mathbf{T}(z_1, z_2) = \frac{\mathbf{P}_{02}z_2^2 + \mathbf{P}_{11}z_1z_2 + \mathbf{P}_{12}z_1z_2^2 + \mathbf{P}_{20}z_1^2 + \mathbf{P}_{21}z_1^2z_2 + \mathbf{P}_{23}z_1^2z_2^3 + \mathbf{P}_{32}z_1^3z_2^2}{-z_2^2 - 2z_1z_2 - 2z_1z_2^2 - z_1^2 + z_1^2z_2 - z_1^2z_2^3 - z_1^3z_2^3 + z_1^4z_2^4}$$

Where,

$$\begin{aligned} \mathbf{P}_{02} &= \begin{bmatrix} 3 & 1 \\ -3 & 1 \end{bmatrix}, \ \mathbf{P}_{11} = \begin{bmatrix} 1 & 3 \\ -13 & -1 \end{bmatrix}, \ \mathbf{P}_{12} = \begin{bmatrix} 0 & -1 \\ 0 & 5 \end{bmatrix} \\ \mathbf{P}_{20} &= \begin{bmatrix} -3 & 0 \\ -11 & -4 \end{bmatrix}, \ \mathbf{P}_{21} = \begin{bmatrix} 3 & 0 \\ -15 & 0 \end{bmatrix} \\ \mathbf{P}_{23} &= \begin{bmatrix} 0 & 1 \\ 6 & 3 \end{bmatrix}, \ \mathbf{P}_{32} = \begin{bmatrix} -3 & 0 \\ -1 & 4 \end{bmatrix} \end{aligned}$$

or

$$\mathbf{T}(z_1, z_2) = \frac{\begin{bmatrix} \alpha_1 & | & \alpha_2 \\ -- & | & -- \\ \alpha_3 & | & \alpha_4 \end{bmatrix}}{-z_2^2 - 2z_1z_2 - 2z_1z_2^2 - z_1^2 + z_1^2z_2 - z_1^2z_2^3 - z_1^3z_2^3 + z_1^4z_2^4} \tag{40}$$

Where,

$$\begin{aligned}
\alpha_1 &= 3z_2^2 + z_1z_2 - 3z_1^2 + 3z_1^2z_2 - 3z_1^3z_2^2 \\
\alpha_2 &= z_2^2 + 3z_1z_2 - z_1z_2^2 + z_1^2z_2^3 \\
\alpha_3 &= -3z_2^2 - 13z_1z_2 - 11z_1^2 - 15z_1^2z_2 + 6z_1^2z_2^3 - z_1^3z_2^2 \\
\alpha_4 &= z_2^2 - z_1z_2 + 5z_1z_2^2 - 4z_1^2 + 3z_1^2z_2^3 + 4z_1^3z_2^2
\end{aligned}$$

Figures 8, 9, 10 and 11 show the mesh, contour, surface and z plots of the above 2D transfer function (40). The $MATLAB^{TM}$ code is given in the Appendix B.

The correctness of the above result (40) can easily be verified using (11), with $k = n = 2$,

$$T(z_1, z_2) = \mathbf{C}\ [\mathbf{I}z_1^2z_2^2 - \mathbf{A}_0z_1z_2 - \mathbf{A}_1z_1 - \mathbf{A}_2z_2]^{-1} \cdot (\mathbf{B}_1z_1 + \mathbf{B}_2z_2).$$

Note–1: It is noted that the structure of the model and the DFT-based, model–to–transfer function, algorithm was illustrated by low order and dimension examples due to simplicity and space requirements.

Note–2: The commas in the subscripts of $a, \mathbf{F}, q, \mathbf{P}$, were ommited from the examples for clarification purposes.

3.5 Complexity of the Algorithm

The proposed algorithm has two parts. In the first part the matrices $\mathbf{F}_{i_1,i_2}$ and the scalars a_{i_1,i_2} are evaluated with a cost of $pmR\lambda^3$ operations. In the second part the coefficients of $\mathbf{P}_{\lambda_1,\lambda_2}$ and q_{λ_1,λ_2} are evaluated using the DFT with a cost of $pmR^2 + R^2$ operations. For more efficient computation, especially for high order systems, fast Fourier methods can be used to implement the DFT [31].

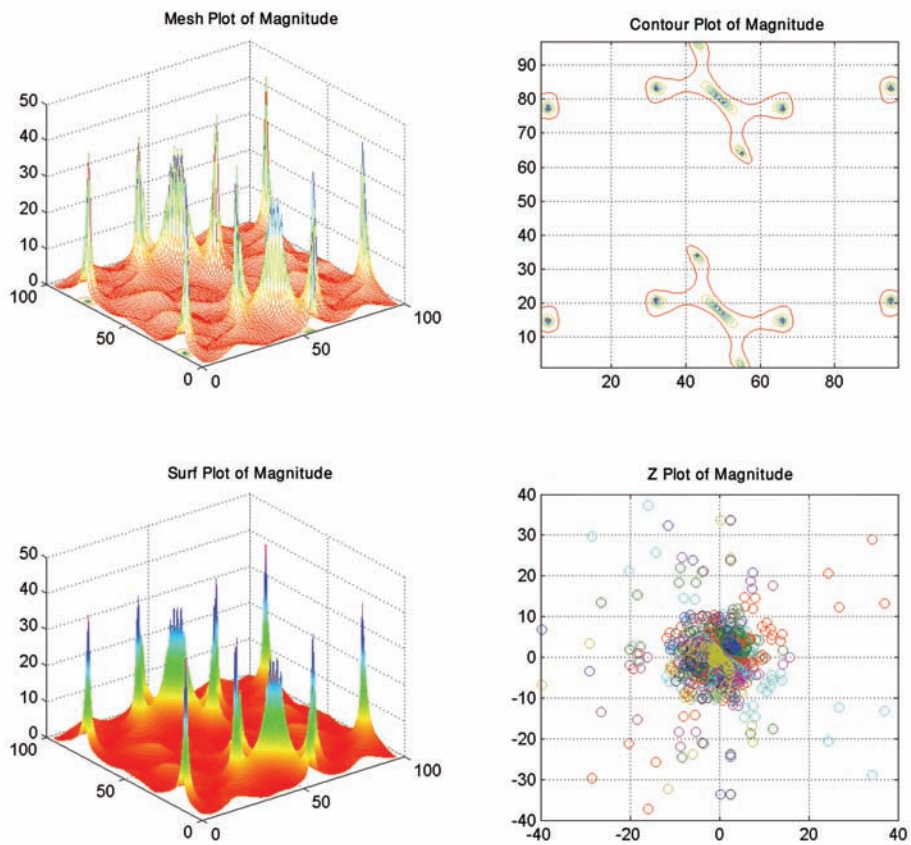

Figure 8: Example 3.4.3 - First Input - First Output

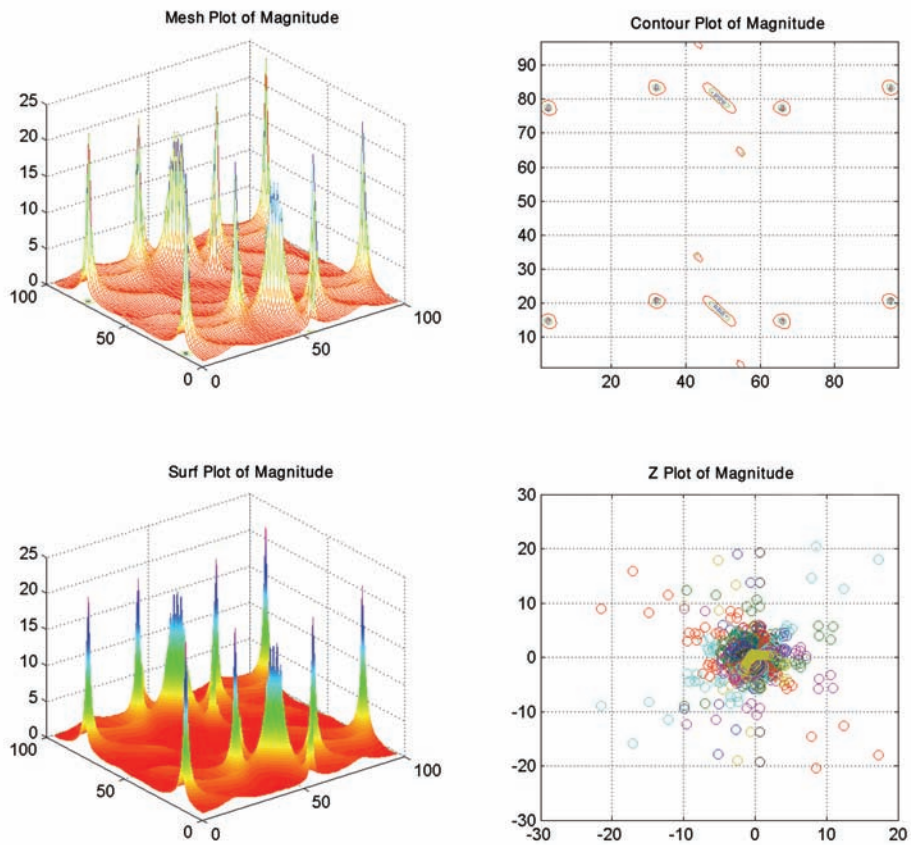

Figure 9: Example 3.4.3 - First Input - Second Output

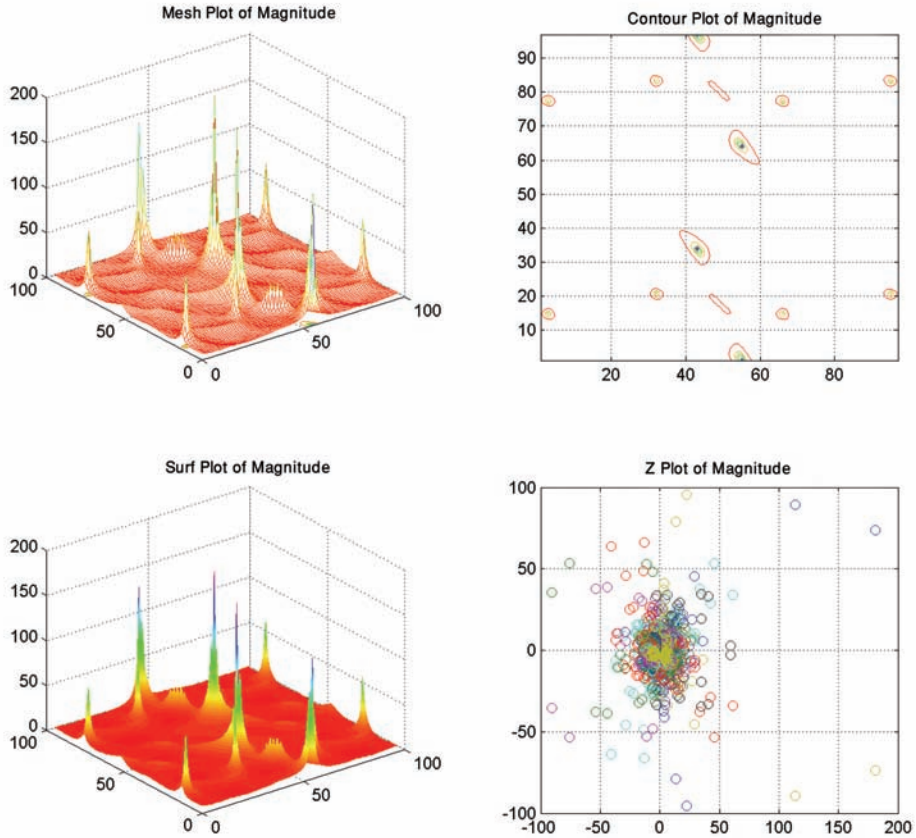

Figure 10: Example 3.4.3 - Second Input - First Output

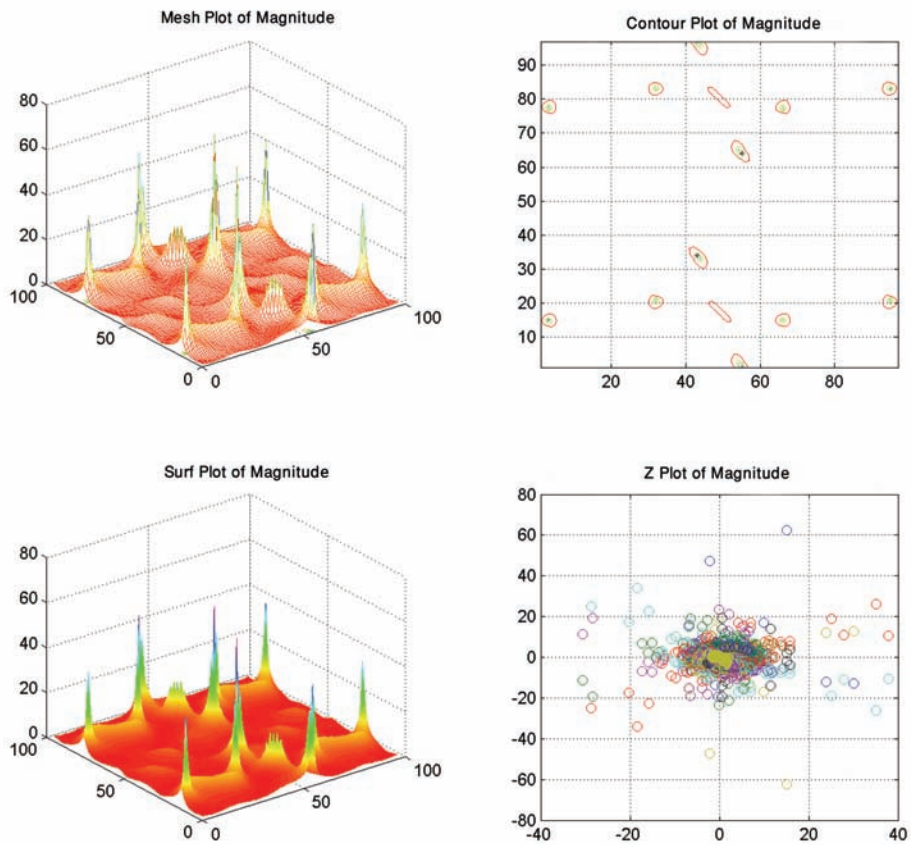

Figure 11: Example 3.4.3 - Second Input - Second Output

Due to the inherent modularity and the algorithmic structure of the presented method high parallelism is permitted. In this case the computation of each determinant a_{i_1,i_2}, (29), and each matrix product $\mathbf{F}_{i_1,i_2}$, (33), can be distributed over a number of processing elements, considerably reducing the computation time of the algorithm.

CHAPTER 4

GENERALIZED SECOND–ORDER TWO-DIMENSIONAL SYSTEM

A generalized second–order two-dimensional (G2O2D) system has the following structure [3]:

$$\begin{aligned} \mathbf{E}x(i_1+2,i_2+2) &= \mathbf{A}_0x(i_1+1,i_2+1)+\mathbf{A}_1x(i_1+1,i_2)+\mathbf{A}_2x(i_1,i_2+1) \\ &\quad +\mathbf{B}_1x(i_1+1,i_2)+\mathbf{B}_2x(i_1,i_2+1) \qquad (41) \\ y(i_1,i_2) &= \mathbf{C}\ x(i_1,i_2) \end{aligned}$$

where, $x(i_1,i_2) \in \mathcal{R}^{\lambda}$, $u(i_1,i_2) \in \mathcal{R}^{m}$, $y(i_1,i_2) \in \mathcal{R}^{p}$; $\mathbf{A}_k$, for $k=1,2$ and $\mathbf{E}, \mathbf{B}_1,\ \mathbf{B}_2,\ \mathbf{C}$, are real matrices of appropriate dimensions. Matrix $\mathbf{E}$ may be singular.

It is noted that this particular S2O2D system (41) is an extension of the 2D Fornasini–Marchesini model [12] to cover systems of second-order. For more 2D second-order structures the reader can refer to [3]. F.L Lewis has given a survey on 1–D and a review on 2–D generalized or singular or implicit systems [23], [24].

Applying the 2D $z_i,\ i=1,2$, transform to system (41), with zero initial conditions, the transfer function is found to be:

$$\mathbf{T}(z_1,z_2)=\mathbf{C}\ [\mathbf{E}z_1^2z_2^2-\mathbf{A}_0z_1z_2-\mathbf{A}_1z_1-\mathbf{A}_2z_2]^{-1}\cdot(\mathbf{B}_1z_1+\mathbf{B}_2z_2) \qquad (42)$$

In the following section an interpolative approach is developed for determining the transfer function $\mathbf{T}(z_1,z_2)$, given the matrices $\mathbf{A}_k$, $k=1,2$ and $\mathbf{B}_1,\ \mathbf{B}_2,\ \mathbf{C}$ using the 2D DFT. For the sake of completeness a brief description of the 2D DFT follows.

4.1 2D Discrete Fourier Transform

Consider the finite sequences $X(k_1, k_2)$ and $\tilde{X}(r_1, r_2)$, $k_i, r_i = 0, \cdots, M_i, \quad \forall\ i = 1, 2$. In order for the sequences $X(k_1, k_2)$ and $\tilde{X}(r_1, r_2)$, to constitute a 2D DFT pair the following relations should hold [10]:

$$\tilde{X}(r_1, r_2) = \sum_{k_1=0}^{M_1} \sum_{k_2=0}^{M_2} X(k_1, k_2) W_1^{-k_1 r_1} W_2^{-k_2 r_2} \tag{43}$$

$$X(k_1, k_2) = \frac{1}{R} \sum_{r_1=0}^{M_1} \sum_{r_2=0}^{M_2} \tilde{X}(r_1, r_2) W_1^{k_1 r_1} W_2^{k_2 r_2} \tag{44}$$

where,

$$R = (M_1 + 1)(M_2 + 1) \tag{45}$$

$$W_i = e^{(2\pi j)/(M_i+1)},\ i = 1, 2 \tag{46}$$

X, $\tilde{X}$ are discrete argument matrix valued functions, with dimensions $p \times m$.

In the following section an interpolative approach is developed for determining the transfer function $T(s)$, given the matrices $\mathbf{A}_i, i = 1, 2$, $\mathbf{E}$, $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{C}$, using the 2D DFT.

4.2 DFT-Based Algorithm

The transfer function of the S2O2D system (41) is,

$$\mathbf{T}(z_1, z_2) = \frac{\mathbf{N}(z_1, z_2}{d(z_1, z_2)} \tag{47}$$

where,

$$\begin{aligned} \mathbf{N}(z_1, z_2) &= \mathbf{C}\ \texttt{adj}\ [\mathbf{E} z_1^2 z_2^2 - \mathbf{A}_0 z_1 z_2 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2] \\ &\quad \times [\mathbf{B}_1 z_1 + \mathbf{B}_2 z_2] \end{aligned} \tag{48}$$

$$d(z_1, z_2) = \texttt{det}\ [\mathbf{E} z_1^2 z_2^2 - \mathbf{A}_0 z_1 z_2 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2] \tag{49}$$

Equations (48) and (49) can be written in polynomial form as follows:

$$\mathbf{N}(z_1, z_2) = \sum_{\lambda_1=0}^{n_{max}^P} \sum_{\lambda_2=0}^{n_{max}^P} \mathbf{P}_{\lambda_1,\lambda_2} z_1^{\lambda_1} z_2^{\lambda_2} \tag{50}$$

with, $n_{max}^P := max(2\lambda)$. The numerator coefficients $\mathbf{P}_{\lambda_1,\lambda_2}$ are matrices with dimensions $(p \times m)$.

$$d(z_1, z_2) = \sum_{\lambda_1=0}^{n_{max}^q} \sum_{\lambda_2=0}^{n_{max}^q} q_{\lambda_1,\lambda_2} z_1^{\lambda_1} z_2^{\lambda_2} \tag{51}$$

where, $n_{max}^q := max(2\lambda)$. The denominator coefficients q_{λ_1,λ_2} are scalars.

The numerator polynomial matrix $\mathbf{N}(z_1, z_2)$ and the denominator polynomial $d(z_1, z_2)$ can be numerically computed at $R = (r+1)^2$, points, equally spaced on the unit 2D disc. The R points are chosen as $(z_1, z_2) = [v(i_1), v(i_2)],\ i_1, i_2 = 0, \cdots, r$, with $r = 2\lambda$, according to definition as:

$$v_1(i) = v_2(i) = W^{-i},\ \ \forall\ \ i = 0, \ldots, r. \tag{52}$$

where,

$$W_i = e^{(2\pi j)/(r+1)},\ \ i = 1, 2 \tag{53}$$

The values of the transfer function (47) at the R points are the corresponding 2D DFT coefficients.

4.2.1 Denominator Polynomial

To evaluate the denominator coefficients q_{λ_1,λ_2}, define,

$$a_{i_1,i_2} = \texttt{det}\ [\mathbf{E} v_1^2(i_1) v_2^2(i_2) - \mathbf{A}_0 v_1(i_1) v_2(i_2) - \mathbf{A}_1 v_1(i_1) - \mathbf{A}_2 v_2(i_2)] \tag{54}$$

Therefore using equations (51) and (54), a_{i_1,i_2} can be defined as,

$$a_{i_1,i_2} = d[v_1(i_1), v_2(i_2)] \tag{55}$$

Provided that at least one of $a_{i_1,i_2} \neq 0$.

Equations (51), (52) and (55) yield

$$a_{i_1,i_2} = \sum_{\lambda_1=0}^{r} \sum_{\lambda_2=0}^{r} q_{\lambda_1,\lambda_2} W^{-(i_1\lambda_1+i_2\lambda_2)} \tag{56}$$

In the above equation (56) it is obvious that $[a_{i_1,i_2}]$ and $[q_{\lambda_1,\lambda_2}]$ form a 2D DFT pair. Therefore the coefficients $[q_{\lambda_1,\lambda_2}]$ can be computed using the inverse 2D DFT, as follows:

$$q_{\lambda_1,\lambda_2} = \frac{1}{R} \sum_{i_1=0}^{r} \sum_{i_2=0}^{r} a_{i_1,i_2} W^{(i_1\lambda_1+i_2\lambda_2)} \tag{57}$$

4.2.2 Numerator Polynomial

To evaluate the numerator matrix polynomial $\mathbf{P}_{\lambda_1,\lambda_2}$, define

$$\begin{aligned} \mathbf{F}_{i_1,i_2} &= \mathbf{C}\ \texttt{adj}\ [\mathbf{E}v_1^2(i_1)v_2^2(i_2) - \mathbf{A}_0 v_1(i_1)v_2(i_2) - \mathbf{A}_1 v_1(i_1) - \mathbf{A}_2 v_2(i_2)] \\ &\quad \times[\mathbf{B}_1 v_1(i_1) + \mathbf{B}_2 v_2(i_2)] \end{aligned} \tag{58}$$

Using equations (50) and (58), $\mathbf{F}_{i_1,i_2}$ can be defined as,

$$\mathbf{F}_{i_1,i_2} = \mathbf{N}[v_1(i_1), v_2(i_2)] \tag{59}$$

Equations (50), (52) and (59) yield

$$\mathbf{F}_{i_1,i_2} = \sum_{\lambda_1=0}^{r-1} \sum_{\lambda_2=0}^{r-1} \mathbf{P}_{\lambda_1,\lambda_2} W^{-(i_1\lambda_1+i_2\lambda_2)} \tag{60}$$

In the above equation (60), $[\mathbf{F}_{i_1,i_2}]$, $[\mathbf{P}_{\lambda_1,\lambda_2}]$ form a 2D DFT pair. Therefore the coefficients $\mathbf{P}_{\lambda_1,\lambda_2}$ can be computed, using the inverse 2D DFT, as follows:

$$\mathbf{P}_{\lambda_1,\lambda_2} = \frac{1}{R} \sum_{i_1=0}^{r-1} \sum_{i_2=0}^{r-1} \mathbf{F}_{i_1,i_2} W^{(i_1\lambda_1+i_2\lambda_2)} \tag{61}$$

Finally, the transfer function sought is,

$$\mathbf{T}(z_1, z_2) = \frac{\mathbf{N}(z_1, z_2)}{d(z_1, z_2)} \tag{62}$$

where,

$$\mathbf{N}(z_1, z_2) = \sum_{\lambda_1=0}^{r-1} \sum_{\lambda_2=0}^{r-1} \mathbf{P}_{\lambda_1,\lambda_2} z_1^{\lambda_1} z_2^{\lambda_2} \tag{63}$$

$$d(z_1, z_2) = \sum_{\lambda_1=0}^{r} \sum_{\lambda_2=0}^{r} q_{\lambda_1,\lambda_2} z_1^{\lambda_1} z_2^{\lambda_2} \tag{64}$$

A salient example, simple yet illustrative of the theoretical concepts presented in this work, follow below:

4.3 Example: G2O2D, two-inputs two-outputs, system

Consider the following generalized second–order 2D system with two–inputs and two–outputs:

$$\begin{aligned} \mathbf{E}x(i_1+2, i_2+2) &= \mathbf{A}_0 x(i_1+1, i_2+1) + \mathbf{A}_1 x(i_1+1, i_2) + \mathbf{A}_2 x(i_1, i_2+1) \\ &\quad + \mathbf{B}_1 u(i_1+1, i_2) + \mathbf{B}_2 u(i_1, i_2+1) \\ y(i_1, i_2) &= \mathbf{C}\, x(i_1, i_2) \end{aligned} \tag{65}$$

where,

$$\begin{aligned} \mathbf{E} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{A}_0 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\ \mathbf{B}_1 &= \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \\ \mathbf{C} &= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \end{aligned}$$

Since $\lambda = 2$, the $r = 2 \times \lambda = 4$. Therefore $R = (r+1)^2 = 25$. The direct application of the proposed algorithm yields:

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} =$$

$$\begin{bmatrix} -7.000+0.000j & 2.736+2.714j & -1.736-2.265j & -1.736-2.265j & 2.736-2.714j \\ -0.+3.804j & 2.045+1.763j & 2.045-0.587j & 1.882+0.812j & -1.309-2.853j \\ 1.618+2.351j & 4.118-3.441j & -3.545-2.853j & -0.191-1.763j & -3.545+0.951j \\ 1.618-2.351j & -3.545-0.951j & -0.191+1.763j & -3.545+0.951j & 4.118+3.441j \\ -0.618-3.804j & -1.309+2.8532j & 1.882-0.812j & 2.045+0.587j & 2.045-1.763j \end{bmatrix}$$

and

$$\begin{aligned}
\mathbf{F}_{00} &= \begin{bmatrix} 1.000 & 4.000 \\ -45.000 & 2.000 \end{bmatrix} \\
\mathbf{F}_{01} &= \begin{bmatrix} -1.763-3.804j & 0.118-2.265j \\ -14.798+30.156j & -9.972-1.987j \end{bmatrix} \\
\mathbf{F}_{02} &= \begin{bmatrix} -6.236-2.351j & -2.118-2.714j \\ 9.798+10.751j & -1.027+5.343j \end{bmatrix} \\
\mathbf{F}_{03} &= \begin{bmatrix} -6.236+2.351j & -2.118+2.714j \\ 9.798-10.751j & -1.027-5.343j \end{bmatrix} \\
\mathbf{F}_{04} &= \begin{bmatrix} -1.763+3.804j & 0.118+2.265j \\ -14.798-30.156j & -9.972+1.987j \end{bmatrix} \\
\mathbf{F}_{10} &= \begin{bmatrix} 5.736-2.714j & 0.809-2.489j \\ 16.444+25.882J & 4.663-2.040j \end{bmatrix}, \\
\mathbf{F}_{11} &= \begin{bmatrix} -6.236+1.175j & -1.427-2.938j \\ 30.978+7.053j & 0.191+5.290j \end{bmatrix}
\end{aligned}$$

$$\mathbf{F}_{12} = \begin{bmatrix} 5.899 + 9.820j & -3.927 + 3.302j \\ 16.281 - 16.531j & 8.545 + 3.302j \end{bmatrix}$$

$$\mathbf{F}_{13} = \begin{bmatrix} 5.736 - 2.9919j & 2.354 + 1.538j \\ -11.972 - 5.898j & -0.500 - 3.441j \end{bmatrix}$$

$$\mathbf{F}_{14} = \begin{bmatrix} 1.000 + 3.526j & 2.191 + 0.587j \\ -7.236 + 21.821j & 3.281 + 8.645j \end{bmatrix}$$

$$\mathbf{F}_{20} = \begin{bmatrix} 1.263 + 2.265j & -0.309 - 0.224j \\ -1.444 - 14.233j & -3.163 - 5.204j \end{bmatrix}$$

$$\mathbf{F}_{21} = \begin{bmatrix} 1.263 - 5.792j & -4.354 - 0.363j \\ -3.027 - 18.102j & -0.500 - 0.812j \end{bmatrix}$$

$$\mathbf{F}_{22} = \begin{bmatrix} -1.763 - 1.902j & 1.927 + 4.755j \\ -15.978 - 11.412j & 1.309 - 8.559j \end{bmatrix}$$

$$\mathbf{F}_{23} = \begin{bmatrix} 1.000 - 5.706j & 3.309 - 0.951j \\ -2.763 + 2.971j & -6.781 - 1.228j \end{bmatrix}$$

$$\mathbf{F}_{24} = \begin{bmatrix} -6.399 - 3.130j & -0.572 - 3.216j \\ 6.218 - 11.531j & 2.954 - 3.216j \end{bmatrix}$$

$$\mathbf{F}_{30} = \begin{bmatrix} 1.263 - 2.265j & -0.309 + 0.224j \\ -1.444 + 14.233j & -3.163 + 5.204j \end{bmatrix}$$

$$\mathbf{F}_{31} = \begin{bmatrix} -6.399 + 3.1307j & -0.572 + 3.216j \\ 6.218 + 11.531j & 2.954 + 3.216j \end{bmatrix}$$

$$\mathbf{F}_{32} = \begin{bmatrix} 1.000 + 5.706j & 3.309 + 0.951j \\ -2.763 - 2.971j & -6.781 + 1.228j \end{bmatrix}$$

$$\mathbf{F}_{33} = \begin{bmatrix} -1.763 + 1.902j & 1.927 - 4.755j \\ -15.978 + 11.412j & 1.309 + 8.559j \end{bmatrix}$$

$$\mathbf{F}_{34} = \begin{bmatrix} 1.263 + 5.792j & -4.354 + 0.363j \\ -3.027 + 18.102j & -0.500 + 0.812j \end{bmatrix}$$

$$
\begin{aligned}
\mathbf{F}_{40} &= \begin{bmatrix} 5.736 + 2.714j & 0.809 + 2.489j \\ 16.444 - 25.882j & 4.663 + 2.040j \end{bmatrix} \\
\mathbf{F}_{41} &= \begin{bmatrix} 1.000 - 3.526j & 2.191 - 0.587j \\ -7.236 - 21.821j & 3.281 - 8.645j \end{bmatrix} \\
\mathbf{F}_{42} &= \begin{bmatrix} 5.736 + 2.991j & 2.354 - 1.538j \\ -11.972 + 5.898j & -0.500 + 3.441j \end{bmatrix} \\
\mathbf{F}_{43} &= \begin{bmatrix} 5.899 - 9.820j & -3.927 - 3.302j \\ 16.281 + 16.531j & 8.545 - 3.302j \end{bmatrix} \\
\mathbf{F}_{44} &= \begin{bmatrix} -6.236 - 1.175j & -1.427 + 2.938j \\ 30.978 - 7.053j & 0.191 - 5.290j \end{bmatrix}
\end{aligned}
$$

Using (57), the denominator coefficients are,

$$
\begin{bmatrix} q_{00} & q_{01} & q_{02} & q_{03} & q_{04} \\ q_{10} & q_{11} & q_{12} & q_{13} & q_{14} \\ q_{20} & q_{21} & q_{22} & q_{23} & q_{24} \\ q_{30} & q_{31} & q_{32} & q_{33} & q_{34} \\ q_{40} & q_{41} & q_{42} & q_{43} & q_{44} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

Using (61), the numerator matrix polynomials are,

$$
\begin{bmatrix} P_{00} & P_{01} & \mathbf{P}_{02} & P_{03} & P_{04} \\ P_{10} & \mathbf{P}_{11} & \mathbf{P}_{12} & P_{13} & P_{14} \\ \mathbf{P}_{20} & \mathbf{P}_{21} & P_{22} & \mathbf{P}_{23} & P_{24} \\ P_{30} & P_{31} & \mathbf{P}_{32} & P_{33} & P_{34} \\ P_{40} & P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix}
$$

where the matrices $P_{00}, \cdots, P_{44}$ are,

$$
\begin{bmatrix}
0 & 0 & \begin{bmatrix} 3 & 1 \\ -3 & 1 \end{bmatrix} & 0 & 0 \\
0 & \begin{bmatrix} 1 & 3 \\ -13 & -1 \end{bmatrix} & \begin{bmatrix} 0 & -1 \\ 0 & 5 \end{bmatrix} & 0 & 0 \\
\begin{bmatrix} -3 & 0 \\ -11 & -4 \end{bmatrix} & \begin{bmatrix} 3 & 0 \\ -15 & 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} & 0 \\
0 & 0 & \begin{bmatrix} -3 & 0 \\ -3 & 0 \end{bmatrix} & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Once the denominator and the adjoint matrix coefficients have been computed, equation (62) can be utilized to obtain the transfer function $\mathbf{T}(z_1, z_2)$. Therefore, we have

$$
\mathbf{T}(z_1, z_2) = \frac{\mathbf{P}_{02}z_2^2 + \mathbf{P}_{11}z_1z_2 + \mathbf{P}_{12}z_1z_2^2 + \mathbf{P}_{20}z_1^2 + \mathbf{P}_{21}z_1^2z_2 + \mathbf{P}_{23}z_1^2z_2^3 + \mathbf{P}_{32}z_1^3z_2^2}{q_{02}z_2^2 + q_{11}z_1z_2 + q_{12}z_1z_2^2 + q_{20}z_1^2 + q_{21}z_1^2z_2 + q_{23}z_1^2z_2^3 + q_{32}z_1^3z_2^2}
$$

or

$$
\mathbf{T}(z_1, z_2) = \frac{\left\{ \begin{array}{l}
\begin{bmatrix} 3 & 1 \\ -3 & 1 \end{bmatrix} z_2^2 + \begin{bmatrix} 1 & 3 \\ -13 & -1 \end{bmatrix} z_1z_2 + \begin{bmatrix} 0 & -1 \\ 0 & 5 \end{bmatrix} z_1z_2^2 \\
+ \begin{bmatrix} -3 & 0 \\ -11 & -4 \end{bmatrix} z_1^2 + \begin{bmatrix} 3 & 0 \\ -15 & 0 \end{bmatrix} z_1^2z_2 + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} z_1^2z_2^3 \\
+ \begin{bmatrix} -3 & 0 \\ -3 & 0 \end{bmatrix} z_1^3z_2^2
\end{array} \right\}}{-z_2^2 - 2z_1z_2 - 2z_1z_2^2 - z_1^2 + z_1^2z_2 - z_1^2z_2^3 - z_1^3z_2^2}
$$

or

$$\mathbf{T}(z_1, z_2) = \frac{\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}}{-z_2^2 - 2z_1z_2 - 2z_1z_2^2 - z_1^2 + z_1^2z_2 - z_1^2z_2^3 - z_1^3z_2^2} \tag{66}$$

where,

$$\begin{aligned}
\alpha_{11} &= 3z_2^2 + z_1z_2 - 3z_1^2 + 3z_1^2z_2 - 3z_1^3z_2^2 \\
\alpha_{12} &= z_2^2 + 3z_1z_2 - z_1z_2^2 + z_1^2z_2^3 \\
\alpha_{21} &= -3z_2^2 - 13z_1z_2 - 11z_1^2 - 15z_1^2z_2 - 3z_1^3z_2^2 \\
\alpha_{22} &= z_2^2 - z_1z_2 + 5z_1z_2^2 - 4z_1^2 + z_1^2z_2^3
\end{aligned}$$

The correctness of the above result (66) can easily be verified using (42).

4.4 Complexity of the Algorithm

The proposed algorithm has two parts. In the first part the matrices $\mathbf{F}_{i_1,i_2}$ and the scalars a_{i_1,i_2} are evaluated with a cost of $pmR\lambda^3$ operations. In the second part the coefficients of $\mathbf{P}_{\lambda_1,\lambda_2}$ and q_{λ_1,λ_2} are evaluated using the DFT with a cost of $pmR^2 + R^2$ operations. For more efficient computation, especially for high order systems, fast Fourier methods can be used to implement the DFT [31].

Due to the inherent modularity and the algorithmic structure of the presented method high parallelism is permitted. In this case the computation of each determinant a_{i_1,i_2}, (56), and each matrix product F_{i_1,i_2}, (60), can be distributed over a number of processing elements, considerably reducing the computation time of the algorithm.

CHAPTER 5

K-ORDER N-DIMENSIONAL SYSTEM

A multi–dimensional (nD) and multi–order k–Order, ($nDkO$), system is described by the following set of equations:

$$
\begin{aligned}
x(i_1+k, i_2+k, i_3+k, \ldots, i_n+k) &= \sum_{\lambda=1}^{k-1} \mathbf{A}_0^{\lambda} x(i_1+\lambda, i_2+\lambda, i_3+\lambda, \ldots, i_n+\lambda) \\
&\quad + \sum_{\mu=1}^{n} \mathbf{A}_{\mu} \underbrace{x(i_1+\nu_1, i_2+\nu_2, i_3+\nu_3, \ldots, i_n+\nu_n)}_{\nu_j = \begin{cases} 1 & for\ \mu = j \\ 0 & for\ \mu \neq j \end{cases}} \\
&\quad + \sum_{\mu=1}^{n} \mathbf{B}_{\mu} \underbrace{u(i_1+\nu_1, i_2+\nu_2, i_3+\nu_3, \ldots, i_n+\nu_n)}_{\nu_j = \begin{cases} 1 & for\ \mu = j \\ 0 & for\ \mu \neq j \end{cases}} \\
y(i_1, i_2, i_3, \ldots, i_n) &= \mathbf{C} x(i_1, i_2, i_3, \ldots, i_n)
\end{aligned}
\tag{67}
$$

where, $x(i_1, i_2, \ldots, i_n) \in \mathcal{R}^{\gamma}$, $u(i_1, i_2, \ldots, i_n) \in \mathcal{R}^{m}$, $y(i_1, i_2, \ldots, i_n) \in \mathcal{R}^{p}$; $\mathbf{A}_0^{\lambda}$, $\lambda = 1, \ldots, k-1$, $\mathbf{A}_{\mu}$, $\mathbf{B}_{\mu}$, $\mu = 1, \ldots, n$, $\mathbf{C}$, are real matrices of appropriate dimensions.

It is noted that the $nDkO$ system (67) is an extension of the 2D model ([4],[9]) to nD and k-Order.

Applying the nD z_i, $(i = 1, 2, \ldots, n)$ transform to the system (67), with zero initial conditions, the corresponding transfer function becomes:

$$
T(z_1, z_2, \ldots, z_n) = \mathbf{C}\, [\mathbf{I} z_1^k z_2^k \cdots z_n^k - \sum_{\lambda=1}^{k-1} \mathbf{A}_0^{\lambda} z_1^{\lambda} z_2^{\lambda} \cdots z_n^{\lambda} - \sum_{\mu=1}^{n} \mathbf{A}_{\mu} z_{\mu}]^{-1} \cdot [\sum_{\mu=1}^{n} \mathbf{B}_{\mu} z_{\mu}] \tag{68}
$$

In the following section an interpolative approach is developed for determining the transfer function $T(z_1, z_2, \ldots, z_n)$, given the matrices, $\mathbf{A}_0^{\lambda}$ for $\lambda = 1, \ldots, k-1$, $\mathbf{A}_{\mu}$, $\mathbf{B}_{\mu}$ for

$\mu = 1, \ldots, n$ and $\mathbf{C}$, using the DFT. For the sake of completeness a brief description of the DFT follows.

5.1 nD DFT

Consider the finite sequences $X(k_1, k_2, \ldots, k_n)$ and $\tilde{X}(r_1,\ r_2, \ldots, r_n)$, $k_i, r_i = 0, \cdots, M_i$, $\forall\ i = 1, 2, \ldots, n$. In order for the sequences $X(k_1, k_2, \ldots, k_n)$ and $\tilde{X}(r_1, r_2,\ \ldots, r_n)$, to constitute a nD DFT pair the following relations should hold [10]:

$$\begin{aligned}
\tilde{X}(r_1, r_2, \ldots, r_n) &= \sum_{k_1=0}^{M_1} \sum_{k_2=0}^{M_2} \cdots \sum_{k_n=0}^{M_n} X(k_1, k_2, \cdots, k_n) \\
&\quad \times W_1^{-k_1 r_1} W_2^{-k_2 r_2} \cdots W_n^{-k_n r_n} \qquad (69) \\
X(k_1, k_2, \ldots, r_n) &= \frac{1}{R} \sum_{r_1=0}^{M_1} \sum_{r_2=0}^{M_2} \cdots \sum_{r_n=0}^{M_n} \tilde{X}(r_1, r_2, \cdots, r_n) \\
&\quad \times W_1^{k_1 r_1} W_2^{k_2 r_2} \cdots W_n^{k_n r_n} \qquad (70)
\end{aligned}$$

where,

$$R = \prod_{i=1}^{n} (M_i + 1) \qquad (71)$$

$$W_i = e^{(2\pi j)/(M_i+1)}, \ \ \forall\ i = 1, 2, \ldots, n \qquad (72)$$

X, $\tilde{X}$ are discrete argument matrix valued functions, with dimensions $p \times m$.

In the following sections an interpolative approach is developed for determining the transfer function $\mathbf{T}(z_1,\ z_2, \ldots, z_n)$, given the matrices $\mathbf{A}_0^\lambda$ for $\lambda = 1, \ldots, k-1$, $\mathbf{A}_\mu$, $\mathbf{B}_\mu$ for $\mu = 1, \ldots, n$ and $\mathbf{C}$, using the DFT.

5.2 Algorithm

The transfer function of the $nDkO$ system (67) has the structure,

$$\mathbf{T}(z_1, z_2, \ldots, z_n) = \frac{\mathbf{N}(z_1, z_2, \ldots, z_n)}{d(z_1, z_2, \ldots, z_n)} \qquad (73)$$

where,

$$\begin{aligned}\mathbf{N}(z_1, z_2, \ldots, z_n) &= \mathbf{C}\ \texttt{adj}\ [\mathbf{I} z_1^k z_2^k \cdots z_n^k - \sum_{\lambda=1}^{k-1} \mathbf{A}_0^\lambda z_1^\lambda z_2^\lambda \cdots z_n^\lambda - \sum_{\mu=1}^{n} \mathbf{A}_\mu z_\mu] \\ &\quad \times \sum_{\mu=1}^{n} \mathbf{B}_\mu z_\mu \qquad (74)\\ d(z_1, z_2, \ldots, z_n) &= \texttt{det}\ [\mathbf{I} z_1^k z_2^k \cdots z_n^k - \sum_{\lambda=1}^{k-1} \mathbf{A}_0^\lambda z_1^\lambda z_2^\lambda \cdots z_n^\lambda - \sum_{\mu=1}^{n} \mathbf{A}_\mu z_\mu] \qquad (75)\end{aligned}$$

Equations (74) and (75) can be written in polynomial form as follows:

$$\mathbf{N}(z_1, z_2, \ldots, z_n) = \underbrace{\sum_{\lambda_1=0}^{n_{max}^{(P)}} \sum_{\lambda_2=0}^{n_{max}^{(P)}} \cdots \sum_{\lambda_n=0}^{n_{max}^{(P)}}}_{\sum_{i=1}^{n} \lambda_i \geq 2n-2} \mathbf{P}_{\lambda_1, \lambda_2, \ldots, \lambda_n} \cdot z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_n^{\lambda_n} \qquad (76)$$

with, $n_{max}^{(P)} := max(n-1)(k-1), (n-1)$. The numerator coefficient matrices, $\mathbf{P}_{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n}$ have dimensions $(p \times m)$.

$$d(z_1, z_2, \ldots, z_n) = \underbrace{\sum_{\lambda_1=0}^{n_{max}^{q}} \sum_{\lambda_2=0}^{n_{max}^{q}} \cdots \sum_{\lambda_n=0}^{n_{max}^{q}}}_{\sum_{i=1}^{n} \lambda_i \geq 2n-1} q_{\lambda_1, \lambda_2, \ldots, \lambda_n} \cdot z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_n^{\lambda_n} \qquad (77)$$

with, $n_{max}^{(q)} := max(n(k-1), n)$. The denominator coefficients $q_{\lambda_1, \lambda_2, \ldots, \lambda_n}$ are scalars.

The numerator polynomial matrix $\mathbf{N}(z_1, z_2, \ldots, z_n)$ and the denominator polynomial $d(z_1, z_2, \ldots, z_n)$ can be numerically computed at $R = \prod_{i=1}^{n}(r+1) = (r+1)^n$ points equally spaced on the unit nD hypersphere, with $r = n \times \gamma$. The R points are chosen as

$$\begin{aligned} v_1(i) &= v_2(i) = \cdots = v_n(i) \\ &= W^{-i},\ \forall\ i = 0, \ldots, r \qquad (78)\end{aligned}$$

where,

$$W_i = W = e^{(2\pi j)/(r+1)},\ \ \forall\ i = 1, 2, \ldots, r \qquad (79)$$

The values of the transfer function at the R points are the corresponding nD DFT coefficients.

5.2.1 Denominator Polynomial

To evaluate the denominator coefficients ($q_{\lambda_1,\lambda_2,\ldots,\lambda_n}$), define,

$$a_{i_1,i_2,\ldots,i_n} = \texttt{det}\ [\mathbf{I}v_1^k(i_1)v_2^k(i_2)\cdots v_n^k(i_n) - \sum_{\lambda=1}^{k-1}\mathbf{A}_0^\lambda v_1^\lambda(i_1)v_2^\lambda(i_2)\times v_n^\lambda(i_n) - \sum_{\mu=1}^{n}\mathbf{A}_\mu v_\mu(i_\mu)] \quad (80)$$

Therefore using equations (77) and (80), yields

$$a_{i_1,i_2,\ldots,i_n} = d[v_1(i_1), v_2(i_2), \cdots, v_n(i_n)] \quad (81)$$

Provided that at least one of $a_{i_1,i_2,\ldots,i_n} \neq 0$.

Equations (77), (78) and (81) yield

$$a_{i_1,i_2,\ldots,i_n} = \sum_{\lambda_1=0}^{r}\sum_{\lambda_2=0}^{r}\cdots\sum_{\lambda_n=0}^{r} q_{\lambda_1,\lambda_2,\ldots,\lambda_n}\cdot W^{-(i_1\lambda_1+i_2\lambda_2+\cdots+i_n\lambda_n)} \quad (82)$$

In the above equation (82) $[a_{i_1,i_2,\ldots,i_n}]$ and $[q_{\lambda_1,\lambda_2,\ldots,\lambda_n}]$ form a DFT pair. Therefore the coefficients $[q_{\lambda_1,\lambda_2,\ldots,\lambda_n}]$ can be computed using the inverse nD DFT, as follows:

$$q_{\lambda_1,\lambda_2,\ldots,\lambda_n} = \frac{1}{R}\sum_{i_1=0}^{r}\sum_{i_2=0}^{r}\cdots\sum_{i_n=0}^{r} a_{i_1,i_2,\ldots,i_n}\cdot W^{(i_1\lambda_1+i_2\lambda_2+\cdots+i_n\lambda_n)} \quad (83)$$

5.2.2 Numerator Polynomial

To evaluate the numerator matrix polynomial $\mathbf{P}_{\lambda_1,\lambda_2,\ldots,\lambda_n}$, define,

$$\begin{aligned}\mathbf{F}_{i_1,i_2,\ldots,i_n} &= \mathbf{C}\ \texttt{adj}\ [\mathbf{I}v_1^k(i_1)v_2^k(i_2)\cdots v_n^k(i_n) - \sum_{\lambda=1}^{k-1}\mathbf{A}_0^\lambda v_1^\lambda(i_1)v_2^\lambda(i_2)\cdots v_n^\lambda(i_n)\\ &\quad - \sum_{\mu=1}^{n}\mathbf{A}_\mu v_\mu(i_\mu)]\cdot[\sum_{\mu=1}^{n}\mathbf{B}_\mu v_\mu(i_\mu)] \quad (84)\end{aligned}$$

Therefore using equations (76) and (84), yields

$$\mathbf{F}_{i_1,i_2,\ldots,i_n} = \mathbf{N}[v_1(i_1), v_2(i_2), \ldots, v_n(i_n)] \quad (85)$$

Equations (76), (78) and (85) yield

$$\mathbf{F}_{i_1,i_2,\ldots,i_n} = \sum_{\lambda_1=0}^{r-1}\sum_{\lambda_2=0}^{r-1}\cdots\sum_{\lambda_n=0}^{r-1}\mathbf{P}_{\lambda_1,\lambda_2,\ldots,\lambda_n}\cdot W^{-(i_1\lambda_1+i_2\lambda_2+\cdots+i_n\lambda_n)} \tag{86}$$

In the above equation (86) it is obvious that $[\mathbf{F}_{i_1,i_2,\ldots,i_n}]$ and $[\mathbf{P}_{\lambda_1,\lambda_2,\ldots,\lambda_n}]$ form a nD DFT pair. Therefore the coefficients $\mathbf{P}_{\lambda_1,\lambda_2,\ldots,\lambda_n}$ can be computed, using the inverse nD DFT, as follows:

$$\mathbf{P}_{\lambda_1,\lambda_2,\ldots,\lambda_n} = \frac{1}{R}\sum_{i_1=0}^{r}\sum_{i_2=0}^{r}\cdots\sum_{i_n=0}^{r}\mathbf{F}_{i_1,i_2,\ldots,i_n}\cdot W^{(i_1\lambda_1+i_2\lambda_2+\cdots+i_n\lambda_n)} \tag{87}$$

Finally, the transfer function sought is,

$$\mathbf{T}(z_1,z_2,\ldots,z_n) = \frac{\mathbf{N}(z_1,z_2,\ldots,z_n)}{d(z_1,z_2,\ldots,z_n)} \tag{88}$$

where,

$$\mathbf{N}(z_1,z_2,\ldots,z_n) = \sum_{\lambda_1=0}^{r-1}\sum_{\lambda_2=0}^{r-1}\cdots\sum_{\lambda_n=0}^{r-1}\mathbf{P}_{\lambda_1,\lambda_2,\ldots,\lambda_n}\cdot z_1^{\lambda_1}z_2^{\lambda_2}\cdots z_n^{\lambda_n} \tag{89}$$

$$d(z_1,z_2,\ldots,z_n) = \sum_{\lambda_1=0}^{r}\sum_{\lambda_2=0}^{r}\cdots\sum_{\lambda_n=0}^{r}q_{\lambda_1,\lambda_2,\ldots,\lambda_n}\cdot z_1^{\lambda_1}z_2^{\lambda_2}\cdots z_n^{\lambda_n} \tag{90}$$

In the next section three illustrative examples are given. The mathematical software package $Matlab^{TM}$ was used for the implementation of the algorithm.

5.3 Complexity of the Algorithm

The proposed algorithm has two parts –In the first part the matrices $\mathbf{F}_{i_1,i_2,\ldots,i_n}$ and the scalars $a_{i_1,i_2,\ldots,i_n}$ are evaluated with a cost of $pmRn^3$ operations. In the second part the coefficients of $\mathbf{P}_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n}$ and $q_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n}$ are evaluated using the DFT with a cost of $pmR^2 + R^2$ operations. For more efficient computation, especially for high order systems, fast Fourier methods can be used to implement the DFT [31].

Due to the inherent modularity and the algorithmic structure of the presented method high parallelism is permitted. In this case the computation of each determinant a_i, (82), and each matrix product F_i, (86), can be distributed over a number of processing elements, considerably reducing the computation time of the algorithm.

CHAPTER 6

GENERALIZED K-ORDER N-DIMENSIONAL SYSTEM

A Generalized (singular) k-Order n-Dimensional ($SkOnD$) model is described by the following set of equations:

$$\begin{aligned}
\mathbf{E}x(i_1+k, i_2+k, i_3+k, \ldots, i_n+k) &= +\sum_{\lambda=1}^{k-1} \mathbf{A}_0^{\lambda} x(i_1+\lambda, i_2+\lambda, i_3+\lambda, \ldots, i_n+\lambda) \\
&\quad + \sum_{\mu=1}^{n} \mathbf{A}_{\mu} \underbrace{x(i_1+\nu_1, i_2+\nu_2, i_3+\nu_3, \ldots, i_n+\nu_n)}_{\nu_j = \begin{cases} 1 & for\ \mu = j \\ 0 & for\ \mu \neq j \end{cases}} \\
&\quad + \sum_{\mu=1}^{n} \mathbf{B}_{\mu} \underbrace{u(i_1+\nu_1, i_2+\nu_2, i_3+\nu_3, \ldots, i_n+\nu_n)}_{\nu_j = \begin{cases} 1 & for\ \mu = j \\ 0 & for\ \mu \neq j \end{cases}} \\
y(i_1, i_2, i_3, \ldots, i_n) &= \mathbf{C}x(i_1, i_2, i_3, \ldots, i_n)
\end{aligned} \tag{91}$$

where, $x(i_1, i_2, i_3, \ldots, i_n) \in \mathcal{R}^{\gamma}$, $u(i_1, i_2, i_3, \ldots, i_n) \in \mathcal{R}^{m}$, $y(i_1, i_2, i_3, \ldots, i_n) \in \mathcal{R}^{p}$; $\mathbf{A}_k$, $k = 1, \ldots, n$, $\mathbf{E}, \mathbf{B}, \mathbf{C}$, are real matrices of appropriate dimensions. matrix $\mathbf{E}$ may be singular with rank ϵ.

It is noted that the $SkOnD$ state space model (91) is an extension of the 2D Fornasini–Marchesini model [3] to n-Dimensions and k-Order. For more 2D second-order model structures the reader can refer to [10].

Applying the nD z_i, $(i = 1, 2, 3, \ldots n)$ transform to the system (91), with zero initial conditions, the transfer function of (91) becomes:

$$T_1(z_1, z_2, z_3, \ldots, z_n) = \mathbf{C}\left[\mathbf{E}z_1^k z_2^k z_3^k \cdots z_n^k - \sum_{\lambda=1}^{k-1} \mathbf{A}_0^{\lambda} z_1^{\lambda} z_2^{\lambda} \cdots z_n^{\lambda} - \sum_{\mu=1}^{n} \mathbf{A}_{\mu} z_{\mu}\right]^{-1}$$

$$\times[\sum_{\mu=1}^{n}\mathbf{B}_{\mu}z_{\mu}] \tag{92}$$

In the following section an interpolative approach is developed for determining the transfer function $T(z_1, z_2, z_3, \ldots, z_n)$, given the matrices $\mathbf{A}_k$ for $k = 1, \ldots, n$ and $\mathbf{E}$, $\mathbf{B}$, $\mathbf{C}$, using the DFT. For the sake of completeness a brief description of the DFT follows.

6.1 nD DFT

Consider the finite sequences $X(k_1, k_2, k_3, \ldots, k_n)$ and $\tilde{X}(r_1, r_2, r_3, \ldots, r_n)$, $k_i, r_i = 0, \cdots, M_i$ $\forall\, i = 1, 2, 3, \ldots, n$. In order for the sequences $X(k_1, k_2, k_3, \ldots, k_n)$ and $\tilde{X}(r_1, r_2, r_3, \ldots, r_n)$, to constitute a nD DFT pair the following relations should hold [10]:

$$\begin{aligned}\tilde{X}(r_1, r_2, r_3, \ldots, r_n) &= \sum_{k_1=0}^{M_1}\sum_{k_2=0}^{M_2}\sum_{k_3=0}^{M_3}\cdots\sum_{k_n=0}^{M_n} X(k_1, k_2, k_3, \cdots, k_n) \\ &\quad \times W_1^{-k_1 r_1} W_2^{-k_2 r_2} W_3^{-k_3 r_3}\cdots W_n^{-k_n r_n}\end{aligned} \tag{93}$$

$$\begin{aligned}X(k_1, k_2, k_3, \ldots, r_n) &= \frac{1}{R}\sum_{r_1=0}^{M_1}\sum_{r_2=0}^{M_2}\sum_{r_3=0}^{M_3}\cdots\sum_{r_n=0}^{M_n} \tilde{X}(r_1, r_2, r_3, \cdots, r_n) \\ &\quad \times W_1^{k_1 r_1} W_2^{k_2 r_2} W_3^{k_3 r_3}\cdots W_n^{k_n r_n}\end{aligned} \tag{94}$$

where,

$$R = \prod_{i=1}^{n}(M_i + 1) \tag{95}$$

$$W_i = e^{(2\pi j)/(M_i+1)}, \;\; \forall\, i = 1, 2, 3, \ldots, n \tag{96}$$

X, $\tilde{X}$ are discrete argument matrix valued functions, with dimensions $p \times m$.

In the following section an interpolative approach is developed for determining the transfer function $\mathbf{T}(z_1, z_2, z_3, \ldots, z_n)$, given the matrices $\mathbf{A}_i, i = 1, \ldots, n$, $\mathbf{E}, \mathbf{B}$ and $\mathbf{C}$, using the DFT.

6.2 Algorithm

The transfer function of the $SkOnD$ state space model (91) has the structure,

$$\mathbf{T}(z_1, z_2, z_3, \ldots, z_n) = \frac{N(z_1, z_2, z_3, \ldots, z_n)}{d(z_1, z_2, z_3, \ldots, z_n)} \tag{97}$$

where,

$$\begin{aligned}
\mathbf{N}(z_1, z_2, z_3, \ldots, z_n) &= \mathbf{C}\ \texttt{adj}\ [\mathbf{E} z_1^k z_2^k z_3^k \cdots z_n^k - \sum_{\lambda=1}^{k-1} \mathbf{A}_0^\lambda z_1^\lambda z_2^\lambda \cdots z_n^\lambda - \sum_{\mu=1}^{n} \mathbf{A}_\mu z_\mu] \\
&\quad \times \sum_{\mu=1}^{n} \mathbf{B}_\mu z_\mu \qquad (98) \\
d(z_1, z_2, z_3, \ldots, z_n) &= \texttt{det}\ [\mathbf{I} z_1^k z_2^k z_3^k \cdots z_n^k - \sum_{\lambda=1}^{k-1} \mathbf{A}_0^\lambda z_1^\lambda z_2^\lambda \cdots z_n^\lambda - \sum_{\mu=1}^{n} \mathbf{A}_\mu z_\mu] \qquad (99)
\end{aligned}$$

It is noted that the (partial) degrees of the denominator are equal to the degree of the system.

Equations (98) and (99) can be written in polynomial form as follows:

$$\mathbf{N}(z_1, z_2, z_3, \ldots, z_n) = \sum_{\lambda_1=0}^{n-1} \sum_{\lambda_2=0}^{n-1} \sum_{\lambda_3=0}^{n-1} \cdots \sum_{\lambda_n=0}^{n-1} \mathbf{P}_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n} z_1^{\lambda_1} z_2^{\lambda_2} z_3^{\lambda_3} \cdots z_n^{\lambda_n} \tag{100}$$

$$d(z_1, z_2, z_3, \ldots, z_n) = \sum_{\lambda_1=0}^{n} \sum_{\lambda_2=0}^{n} \sum_{\lambda_3=0}^{n} \cdots \sum_{\lambda_n=0}^{n} q_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n} z_1^{\lambda_1} z_2^{\lambda_2} z_3^{\lambda_3} \cdots z_n^{\lambda_n} \tag{101}$$

$\mathbf{P}_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n}$ are matrices with dimensions $(p \times m)$, while $q_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n}$ are scalars.

The numerator polynomial matrix $\mathbf{N}(z_1, z_2, z_3, \ldots, z_n)$ and the denominator polynomial $d(z_1, z_2, z_3, \ldots, z_n)$ can be numerically computed at $R = \prod_{i=1}^{n}(r+1) = (r+1)^n$ points equally spaced on the unit nD hypersphere, with $r = n \times \gamma$. The R points are chosen as

$$v_1(i) = v_2(i) = v_3(i) = \cdots = v_n(i) = W^{-i}, \ \ \forall \ \ i = 0, \ldots, r \tag{102}$$

where,

$$W_i = W = e^{(2\pi j)/(r+1)}, \ \ \forall\, i = 1, 2, 3, \ldots, r \tag{103}$$

The values of the transfer function (97) at the R points are the corresponding nD DFT coefficients.

6.2.1 Denominator Polynomial

To evaluate the denominator coefficients ($q_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n}$), define,

$$\begin{aligned} a_{i_1,i_2,i_3,\ldots,i_n} &= \texttt{det}\,[\mathbf{E}v_1^k(i_1)v_2^k(i_2)v_3^k(i_3)\cdots v_n^k(i_n) \\ &\quad - \sum_{\lambda=1}^{k-1}\mathbf{A}_0^\lambda v_1^\lambda(i_1)v_2^\lambda(i_2)v_3^\lambda(i_3)\cdots v_n^\lambda(i_n) \\ &\quad - \sum_{\mu=1}^{n}\mathbf{A}_\mu v_\mu(i_\mu)] \end{aligned} \tag{104}$$

Therefore using equations (101) and (104) yields

$$a_{i_1,i_2,i_3,\ldots,i_n} = d[v_1(i_1), v_2(i_2), v_3(i_3), \cdots, v_n(i_n)] \tag{105}$$

Provided that at least one of $a_{i_1,i_2,i_3,\ldots,i_n} \neq 0$.

Equations (101), (102) and (105) yield

$$a_{i_1,i_2,i_3,\ldots,i_n} = \sum_{\lambda_1=0}^{r}\sum_{\lambda_2=0}^{r}\sum_{\lambda_3=0}^{r}\cdots\sum_{\lambda_n=0}^{r} q_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n} W^{-(i_1\lambda_1+i_2\lambda_2+i_3\lambda_3+\cdots+i_n\lambda_n)} \tag{106}$$

In the above equation (106) $[a_{i_1,i_2,i_3,\ldots,i_n}]$ and $[q_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n}]$ form a DFT pair. Therefore the coefficients $[q_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n}]$ can be computed using the inverse nD DFT, as follows:

$$q_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n} = \frac{1}{R}\sum_{i_1=0}^{r}\sum_{i_2=0}^{r}\sum_{i_3=0}^{r}\cdots\sum_{i_n=0}^{r} a_{i_1,i_2,i_3,\ldots,i_n} W^{(i_1\lambda_1+i_2\lambda_2+i_3\lambda_3+\cdots+i_n\lambda_n)} \tag{107}$$

6.2.2 Numerator Polynomial

To evaluate the numerator matrix polynomial $\mathbf{P}_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n}$, define,

$$\begin{aligned} \mathbf{F}_{i_1,i_2,i_3,\ldots,i_n} &= \mathbf{C}\ \texttt{adj}\,[\mathbf{E}v_1^k(i_1)v_2^k(i_2)v_3^k(i_3)\cdots v_n^k(i_n) \\ &\quad - \sum_{\lambda=1}^{k-1}\mathbf{A}_0^\lambda v_1^\lambda(i_1)v_2^\lambda(i_2)v_3^\lambda(i_3)\cdots v_n^\lambda(i_n) \\ &\quad - \sum_{\mu=1}^{n}\mathbf{A}_\mu v_\mu(i_\mu)] \times [\sum_{\mu=1}^{n}\mathbf{B}_\mu v_\mu(i_\mu)] \end{aligned} \tag{108}$$

Therefore using equations (100) and (108), yields

$$\mathbf{F}_{i_1,i_2,i_3,\ldots,i_n} = \mathbf{N}[v_1(i_1), v_2(i_2), v_3(i_3), \ldots, v_n(i_n)] \tag{109}$$

Equations (100), (102) and (109) yield

$$\mathbf{F}_{i_1,i_2,i_3,\ldots,i_n} = \sum_{\lambda_1=0}^{r-1}\sum_{\lambda_2=0}^{r-1}\sum_{\lambda_3=0}^{r-1}\cdots\sum_{\lambda_n=0}^{r-1}\mathbf{P}_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n}W^{-(i_1\lambda_1+i_2\lambda_2+i_3\lambda_3+\cdots+i_n\lambda_n)} \tag{110}$$

In the above equation (110) it is obvious that $[\mathbf{F}_{i_1,i_2,i_3,\ldots,i_n}]$ and $[\mathbf{P}_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n}]$ form a nD DFT pair. Therefore the coefficients $\mathbf{P}_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n}$ can be computed, using the inverse nD DFT, as follows:

$$\mathbf{P}_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n} = \frac{1}{R}\sum_{i_1=0}^{r}\sum_{i_2=0}^{r}\sum_{i_3=0}^{r}\cdots\sum_{i_n=0}^{r}\mathbf{F}_{i_1,i_2,i_3,\ldots,i_n}W^{(i_1\lambda_1+i_2\lambda_2+i_3\lambda_3+\cdots+i_n\lambda_n)} \tag{111}$$

Finally, the transfer function sought is,

$$\mathbf{T}(z_1, z_2, z_3, \ldots, z_n) = \frac{N(z_1, z_2, z_3, \ldots, z_n)}{d(z_1, z_2, z_3, \ldots, z_n)} \tag{112}$$

where,

$$\mathbf{N}(z_1, z_2, z_3, \ldots, z_n) = \sum_{\lambda_1=0}^{r-1}\sum_{\lambda_2=0}^{r-1}\sum_{\lambda_3=0}^{r-1}\cdots\sum_{\lambda_n=0}^{r-1}\mathbf{P}_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n}z_1^{\lambda_1}z_2^{\lambda_2}z_3^{\lambda_3}\cdots z_n^{\lambda_n} \tag{113}$$

$$d(z_1, z_2, z_3, \ldots, z_n) = \sum_{\lambda_1=0}^{r}\sum_{\lambda_2=0}^{r}\sum_{\lambda_3=0}^{r}\cdots\sum_{\lambda_n=0}^{r}q_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n}z_1^{\lambda_1}z_2^{\lambda_2}z_3^{\lambda_3}\cdots z_n^{\lambda_n} \tag{114}$$

6.3 Complexity of the Algorithm

The proposed algorithm has two parts. In the first part the matrices $\mathbf{F}_{i_1,i_2,\ldots,i_n}$ and the scalars $a_{i_1,i_2,\ldots,i_n}$ are evaluated with a cost of $pmRn^3$ operations. In the second part the coefficients of $\mathbf{P}_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n}$ and $q_{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n}$ are evaluated using the DFT with a cost of $pmR^2 + R^2$ operations. For more efficient computation, especially for high order systems, fast Fourier methods can be used to implement the DFT [31].

Due to the inherent modularity and the algorithmic structure of the presented method high parallelism is permitted. In this case the computation of each determinant a_i, (104), and each matrix product F_i, (108), can be distributed over a number of processing elements, considerably reducing the computation time of the algorithm.

CHAPTER 7

CONCLUSION

In this thesis the DFT was used for computing the transfer functions for the following systems/models:

- Second–order 2D system
- Generalized second–order 2D system
- k–order n–dimensional system
- Generalized k–order n–dimensional system

The algorithms were implemented with the software package $Matlab^{TM}$. To further improve the computational speed of the algorithm, fast Fourier techniques and computer systems with multiple central processing units can be used.

Moreover, two second–order 2D models were extended to n–dimensional k–order (multi–dimensional multi–order) for regular and generalized systems.

As it can be seen from the 2D second–order examples the presented models can realize 2D transfer functions in a more compact form having lower matrix–dimensions, (2×2), than the classical 2D models that require (4×4) matrices. This is also the case for higher dimension–order systems.

CHAPTER 8

FUTURE WORK

Using the regular or generalized 2(k)–order 2(n)–D models the following problems can be considered.

- Stability,
- Solvability and properties,
- Reachability and observability,
- Model equivalence,
- Regularity,
- Geometric theory,
- Minimum energy control,
- More (faster) techniques for transfer function computation,
- State–space and circuit realization etc.,
- Positive k-order nD system–based problems
- VHDL/FPGA implementations
- Applications to filtering

APPENDIX A

$MATLAB^{TM}$ CODE-1

MATLAB^{TM} code for computing the transfer function coeffients using the 2D DFT.

```
E  =  [1 0; 0 1]

A0 =  [1 3; 0 0]

A1 =  [-1 1; 0  1]

A2 =  [0 1; 1  1]

B1 =  [1 ; 0]

B2 =  [0 ; 1]

C  =  [0  1]

D  =  0

%%-----------------------------------------------------------------------%%


I = sqrt(-1)


N = 2;


Wtemp = exp(2*pi*I/5);

clear Z

Z(1,1) = 1;


Z(1,2) = Wtemp^(-1);


Z(1,3) = Wtemp^(-2);
```

```
Z(1,4) = Wtemp^(-3);

Z(1,5) = Wtemp^(-4);

W(1,1) = Z(1,1);

W(1,2) = Z(1,2);

W(1,3) = Z(1,3);

W(1,4) = Z(1,4);

W(1,5) = Z(1,5);



%%-----------------------------------------------------------------------%%
%%    Calculation of                                                     %%
%%    E*Z(I)*W(J) + A1 A2 - A1 Z(I) - A2 W(J)                            %%
%%    For all (I,J) <= (2,2)                                             %%
%%-----------------------------------------------------------------------%%
for i=1:5
    for j=1:5
      K{i,j}=E*Z(i)*Z(i)*W(j)*W(j)-A0*Z(i)*W(j)-A1*Z(i)-A2*W(j);
  end
end

for i=1:5
    for j=1:5
```

```
        str=strcat('K(',num2str(i-1),',',num2str(j-1),')=');
        disp(str)
        disp(K{i,j}) %Display K
    end

end
%%-------------------------------------------------------------------------%%
%%    Calculation of                                                       %%
%%    A(IJ)                                                                %%
%%-------------------------------------------------------------------------%%
for i=1:5
    for j=1:5
       A(i,j)=det(K{i,j});

    end
end
%GA=num2cell(A)

%%-------------------------------------------------------------------------%%
%%    Calculation of                                                       %%
%%    F(IJ)                                                                %%
%%-------------------------------------------------------------------------%%

for i=1:5
    for j=1:5

    F(i,j)=C*inv(K{i,j})*A(i,j)*(B1*Z(i)+B2*W(j));

    end
```

```
end



%%-----------------------------------------------------------------------%%
%%    Calculation of                                                     %%
%%    P(IJ)                                                              %%
%%-----------------------------------------------------------------------%%

P=zeros(5,5);
for lambda1=1:5
    for lambda2=1:5


        for i=1:5
            for j=1:5
                P(lambda1,lambda2)= P(lambda1,lambda2)...;
                 .+ F(i,j)*Wtemp^((i-1)*(lambda1-1)+(j-1)*(lambda2-1));
            end
        end
      P(lambda1,lambda2)=round(P(lambda1,lambda2))/25;



    end
end



%%-----------------------------------------------------------------------%%
%%    Calculation of                                                     %%
%%    Q(IJ)                                                              %%
%%-----------------------------------------------------------------------%%
```

```
Q=zeros(5,5);
for lambda1=1:5
    for lambda2=1:5


        for i=1:5
            for j=1:5
                Q(lambda1,lambda2)= Q(lambda1,lambda2)...;
                 .+ A(i,j)*Wtemp^((i-1)*(lambda1-1)+(j-1)*(lambda2-1));
            end
        end
      Q(lambda1,lambda2)=round(Q(lambda1,lambda2))/25;




    end


end


A %Display A
F %Display F
P %Display Numerator Polynomial
Q %Display Denominator Polynomial
```

```
%%-----------------------------------------------------------------------%%
%% Calculating the Transfer Function using formula:                      %%
%% T(z1,z2)=C*((I*(z1^2)*(z2^2)-A0*z1*z2-A1*z1-A2*z2)^-1)                %%
%%          *(B1*z1+B2*z2)                                               %%
%%                                                                       %%
%%-----------------------------------------------------------------------%%
for z1=1:5
    for z2=1:5
        Transfer(z1,z2)=C*((I*(z1^2)*(z2^2)-A0*z1*z2-A1*z1-A2*z2)^-1)...;
                        .*(B1*z1+B2*z2);
    end
end

Transfer %Display Transfer function



%%-----------------------------------------------------------------------%%
%% Graphing the A matrice                                                %%
%%-----------------------------------------------------------------------%%

ww1 = -6.8;
ww2 =  6.8;
step = 0.1;
[w1,w2]=meshgrid(ww1:step:ww2, ww1:step:ww2);
plot(real(A), imag(A))
subplot(2,2,1)
meshc(A)
axis square
shading interp
```

```
title ('Fig X.XX transfer function mesh plot with contours')
subplot(2,2,2)
contour(A)
axis square
shading interp
title ('Fig X.XX transfer function contour plot')
subplot(2,2,3)
surf(A)
axis square
shading interp
title ('Fig X.XX transfer function surf plot')
subplot(2,2,4)
contourf(A,30)
axis square
title ('Fig X.XX transfer function contour plot of peaks')
pause(4)

%%-----------------------------------------------------------------------%%
% Graphing the F matrice                                                 %%
%%-----------------------------------------------------------------------%%

ww1 = -6.8;
ww2 =  6.8;
step = 0.1;
[w1,w2]=meshgrid(ww1:step:ww2, ww1:step:ww2);
plot(real(F), imag(F))
subplot(2,2,1)
meshc(F)
axis square
```

```
shading interp
title ('Fig X.XX transfer function mesh plot with contours')
subplot(2,2,2)
contour(F)
axis square
shading interp
title ('Fig X.XX transfer function contour plot')
subplot(2,2,3)
surf(F)
axis square
shading interp
title ('Fig X.XX transfer function surf plot')
subplot(2,2,4)
contourf(F,30)
axis square
title ('Fig X.XX transfer function contour plot of peaks')

%%-----------------------------------------------------------------------%%
%%-----------------------------------------------------------------------%%
```

APPENDIX B

$MATLAB^{TM}$ CODE-2

MATLAB^{TM} code for the figures in the thesis.

```
clear;
pack;
clg;

ww1 = -4.8;
ww2 =  4.8;
step = 0.1;
[w1,w2]=meshgrid(ww1:step:ww2, ww1:step:ww2);
j=sqrt(-1);
z1=exp(j*w1);
z2=exp(j*w2);
%z_A= (z1.*z2+2*z2)...;
%./(z1.*z1.*z1.*z1.*z2.*z2.*z2.*z2 + 3*z1.*z1.*z2  + 6*z1.*z2);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Example-1 data with 1 input and 1 output                                %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
z_A= (z1.*z1.*z2.*z2.*z2 - z1.*z2.*z2 + 2*z1.*z2)...;
   ./(z1.*z1.*z1.*z1.*z2.*z2.*z2.*z2 - z1.*z1.*z1.*z2.*z2.*z2
    - z1.*z1.*z2.*z2.*z2  + z1.*z1.*z2 - z1.*z1 - 2*z1.*z2.*z2
    - 2*z1.*z2 - z2.*z2);
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Example-2 data with 2 input and 1 output                                %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
z_B_input1= (z1.*z1.*z1.*z2.*z2 + 3*z1.*z1.*z2.*z2.*z2 - 9*z1.*z1.*z2
            - 4*z1.*z1 - 7*z1.*z2 - 3*z2.*z2)...;
          ./(z1.*z1.*z1.*z1.*z2.*z2.*z2.*z2 - z1.*z1.*z1.*z2.*z2.*z2
            - z1.*z1.*z2.*z2.*z2  + z1.*z1.*z2 - z1.*z1 - 2*z1.*z2.*z2
            - 2*z1.*z2 - z2.*z2);
z_B_input2= (2*z1.*z1.*z1.*z2.*z2 + z1.*z1.*z2.*z2.*z2 - 2*z1.*z1
            + 3*z1.*z2.*z2 - 2*z1.*z2)...;
          ./(z1.*z1.*z1.*z1.*z2.*z2.*z2.*z2 - z1.*z1.*z1.*z2.*z2.*z2
            - z1.*z1.*z2.*z2.*z2  + z1.*z1.*z2 - z1.*z1 - 2*z1.*z2.*z2
            - 2*z1.*z2 - z2.*z2);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Example-3 data with 2 input and 2 output                                %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
z_C_input1_output1= (3*z2.*z2 + z1.*z2 - 3*z1.*z1 + 3*z1.*z1.*z2
            - 3*z1.*z1.*z1.*z2.*z2)...;
           ./(z1.*z1.*z1.*z1.*z2.*z2.*z2.*z2 - z1.*z1.*z1.*z2.*z2.*z2
            - z1.*z1.*z2.*z2.*z2  + z1.*z1.*z2 - z1.*z1 - 2*z1.*z2.*z2
            - 2*z1.*z2 - z2.*z2);
z_C_input1_output2= (z2.*z2 + 3*z1.*z2 - z1.*z2.*z2
            + z1.*z1.*z2.*z2.*z2)...;
           ./(z1.*z1.*z1.*z1.*z2.*z2.*z2.*z2 - z1.*z1.*z1.*z2.*z2.*z2
            - z1.*z1.*z2.*z2.*z2  + z1.*z1.*z2 - z1.*z1 - 2*z1.*z2.*z2
            - 2*z1.*z2 - z2.*z2);
z_C_input2_output1= (-3*z2.*z2 - 13*z1.*z2 - 11*z1.*z1 - 15.*z1.*z1.*z2
            + 6*z1.*z1.*z2.*z2.*z2 - z1.*z1.*z1.*z2.*z2)...;
           ./(z1.*z1.*z1.*z1.*z2.*z2.*z2.*z2 - z1.*z1.*z1.*z2.*z2.*z2
```

```
            - z1.*z1.*z2.*z2.*z2  + z1.*z1.*z2 - z1.*z1 - 2*z1.*z2.*z2
            - 2*z1.*z2 - z2.*z2);
z_C_input2_output2= (z2.*z2-z1.*z2 + 5*z1.*z2.*z2 - 4*z1.*z1
            + 3*z1.*z1.*z2.*z2.*z2 + 4*z1.*z1.*z1.*z2.*z2)...;
           ./(z1.*z1.*z1.*z1.*z2.*z2.*z2.*z2 - z1.*z1.*z1.*z2.*z2.*z2
            - z1.*z1.*z2.*z2.*z2  + z1.*z1.*z2 - z1.*z1 - 2*z1.*z2.*z2
            - 2*z1.*z2 - z2.*z2);

%%%
```

```
% plot(real(z1), imag(z1),'>', real(z2'),imag(z2'),'+')
pause(4)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plotting the transfer function of the two-Input Single-Output example  %
% Plotting: Input1                                                        %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
subplot(2,2,1), meshc(abs(z_B_input1));
grid on;
title('ex. B Input 1 - Original magnitude ');

subplot(2,2,2),contour(abs(z_B_input1));
grid on;
title('ex. B Input 1 - Contour of original mag.');

subplot(2,2,3),surf(abs(z_B_input1),'facecolor','black','edgecolor',...;
    'yellow');
grid on;
colormap hsv
shading interp
title('ex. B Input 1 - surface of original mag.');

subplot(2,2,4), plot(real(z_B_input1), imag(z_B_input1), 'o');
grid on;
title('ex. B Input 1 - z plot of original mag.');
pause(4)
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plotting the transfer function of the two-Input Single-Output example  %
% Plotting: Input2                                                       %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
subplot(2,2,1), meshc(abs(z_B_input2));
grid on;
title('ex. B Input 2 - Original magnitude ');

subplot(2,2,2),contour(abs(z_B_input2));
grid on;
title('ex. B Input 2 - Contour of original mag.');

subplot(2,2,3), surf(abs(z_B_input2),'facecolor','black','edgecolor',...;
                    .'yellow');
grid on;
colormap hsv
shading interp
title('ex. B Input 2 - surface of original mag.');

subplot(2,2,4), plot(real(z_B_input2), imag(z_B_input2), 'o');
grid on;
title('ex. B Input 2 - z plot of original mag.');
pause(4)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plotting the transfer function of the Two-Input Two-Output example     %
% Plotting: Input1 - Output1                                             %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
subplot(2,2,1), meshc(abs(z_C_input1_output1));
grid on;
```

```
title('ex. C Input 1 - Output 1 - Original magnitude ');

subplot(2,2,2),contour(abs(z_C_input1_output1));
grid on;
title('ex. C Input 1 - Output 1 - Contour of original mag.');

subplot(2,2,3),surf(abs(z_C_input1_output1),'facecolor','black',...;
                   .'edgecolor','yellow');
grid on;
colormap hsv
shading interp
title('ex. C Input 1 - Output 1 - surface of original mag.');

subplot(2,2,4),plot(real(z_C_input1_output1),imag(z_C_input1_output1),'o');
grid on;
title('ex. C Input 1 - Output 1 - z plot of original mag.');
pause(4)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plotting the transfer function of the Two-Input Two-Output example     %
% Plotting: Input1 - Output2                                             %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
subplot(2,2,1), meshc(abs(z_C_input1_output2));
grid on;
title('ex. C Input 1 - Output 2 - Original magnitude ');

subplot(2,2,2),contour(abs(z_C_input1_output2));
grid on;
title('ex. C Input 1 - Output 2 - Contour of original mag.');
```

```
subplot(2,2,3), surf(abs(z_C_input1_output2), 'facecolor', 'black',...;
                    .'edgecolor', 'yellow');
grid on;
colormap hsv
shading interp
title('ex. C Input 1 - Output 2 - surface of original mag.');

subplot(2,2,4),plot(real(z_C_input1_output2),imag(z_C_input1_output2),'o');
grid on;
title('ex. C Input 1 - Output 2 - z plot of original mag.');
pause(4)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plotting the transfer function of the Two-Input Two-Output example      %
% Plotting: Input2 - Output1                                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
subplot(2,2,1), meshc(abs(z_C_input2_output1));
grid on;
title('ex. C Input 2 - Output 1 - Original magnitude ');

subplot(2,2,2),contour(abs(z_C_input2_output1));
grid on;
title('ex. C Input 2 - Output 1 - Contour of original mag.');

subplot(2,2,3), surf(abs(z_C_input2_output1), 'facecolor', 'black',...;
                    .'edgecolor', 'yellow');
grid on;
colormap hsv
shading interp
title('ex. C Input 2 - Output 1 - surface of original mag.');
```

```
subplot(2,2,4),plot(real(z_C_input2_output1),imag(z_C_input2_output1),'o');
grid on;
title('ex. C Input 2 - Output 1 - z plot of original mag.');
pause(4)
%%%
```

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VITA

Marinos T. Michael was born in Limassol, Cyprus. He was awarded his degree of Bachelor of Science in Computer Science (Magna Cum Laude) in 2004 and the degree of Master of Science in Computer Science in 2006 from Montclair State University, Montclair, New Jersey, USA. He is a student member of the IEEE.